INTERACTIONS BETWEEN ELECTRON BEAMS AND FULLY IONIZED PLASMAS

STANLEY D. COX

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by

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Captain, United States Marine Corps

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the requirements for the degree of

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ABSTRACT

Interaction between an electron beam and a fully ionized plasma has been studied with a view towards its application in a structure-less traveling wave tube. Three basic approaches, of varying degree of rigor, to the problem have been pursued and analytical solutions for the circularly symmetric case obtained. Comparisons between the methods of analysis are made.

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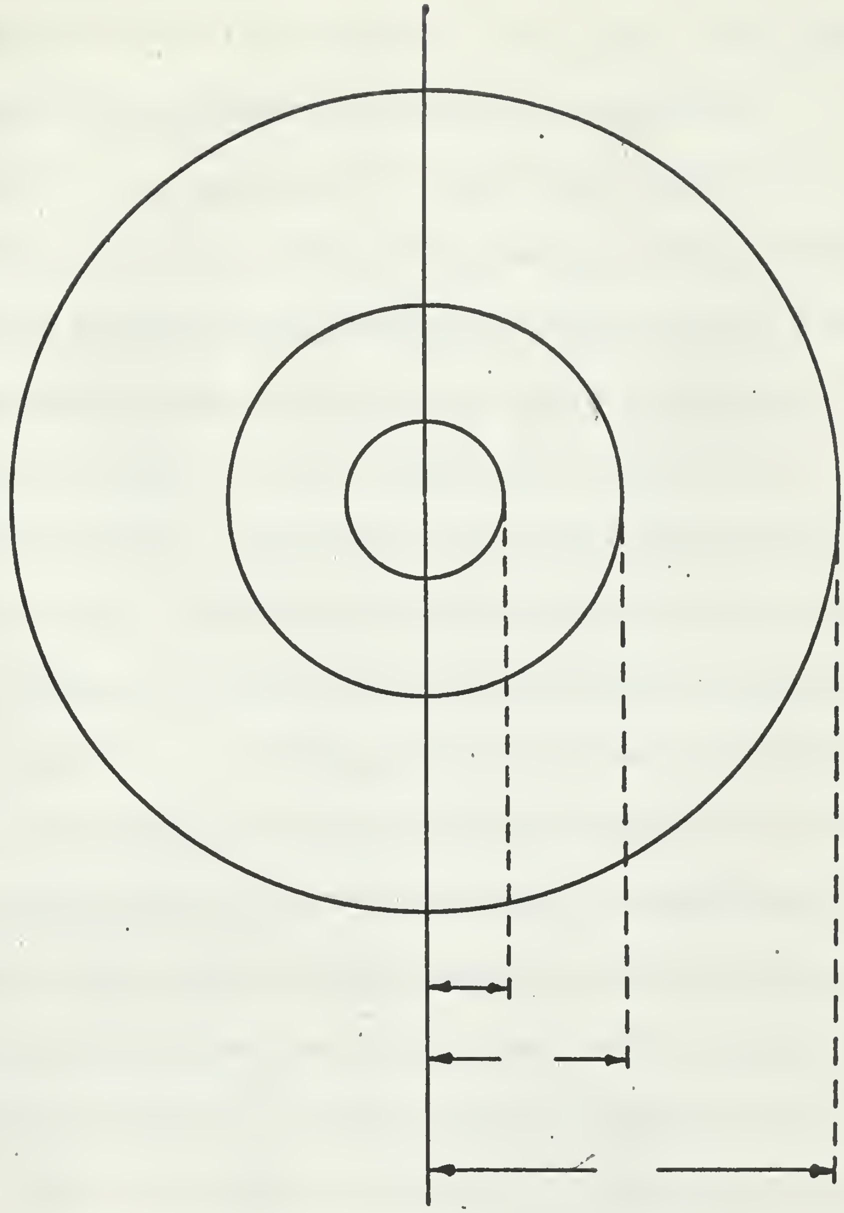
1. Introduction

Investigations into wave propagation in plasmas have been conducted by a number of workers (1), (2), (3) with a view towards employment in plasma diagnostics and traveling wave tubes. Extensive bibliographies are contained in these works and the reader is referred to these for previous work in the field.

Basically, the problem is a boundary value problem in cylindrical co-ordinates (r,θ,z) as depicted in Fig. 1 with a static magnetic field parallel to the z-axis. Throughout this paper, the case of coupling to a helix will be considered as this is a common method of coupling into or out of a traveling wave tube. The mathematical model used for the helix is the simplest of those that have been developed, that of the "sheath helix". The reader is referred to any standard work, such as Pierce (4) for details on this point. The results will be used without comment in this paper.

The mathematical model used for the plasma will be that customarily used in studies of this nature unless specifically stated otherwise. This is the "cold" plasma wherein effects of collisions, recombinations, neutrals, and thermal motion are ignored and the electrons are considered as forming a "cloud" against a background of positive ions which provide overall (DC) neutrality of charge, but do not otherwise appreciably contribute to the problem due to the relatively large mass of the ions which renders them practically stationary.

Three approaches to the problem are presented and compared. These



$0 < r < a$	Electron Beam
$a < r < b$	Plasma
$b < r < c$	Free Space
$r = c$	Helix
$r > c$	Free Space

Figure 1
Geometry of the Problem

are presented in order of increasing complexity, if not rigor, and the results are compared using the third method as the standard of comparison

The first method is an extension of the works of Boyd, Gould and Trivelpiece and has, as the major distinguishing features, the assumptions that the magnitude of the longitudinal propagation constant is much greater than that of free space and that the electric field may be represented as derived from a scalar potential. This is Trivelpiece's "slow wave" or "quasi-static" approximation. This method has the advantage of simplicity albeit at the price of rigor, but the extreme simplicity alone is of considerable practical value provided, of course, that the results provide a reasonable approximation to the true case. A feature of this "slow wave" approximation is that a TE mode of propagation is denied by the first assumptions. While of no great consequence when dealing with drift tubes or waveguides, this is troublesome when one attempts to derive expressions for interactions with a helix. Matching boundary conditions at the "sheath helix" requires both TE and TM types of solutions as is stated in Hutter (5) and can quickly be demonstrated. Thus, a dilemma presents itself. In attempting to extend this simple method, a free space TE solution will be assumed within the electron beam and plasma regions and the inconsistency ignored.

The second method is an extension of the works of Rigrod and Lewis (6), and Brewer (7), the latter being essentially a generalization of the former. This method, as developed for electron beam studies, solves Maxwell's equations in a region containing charge, and, through a perturbational

approach, takes the effect of the charge into consideration in the boundary value problem by replacing the rippled beam by an equivalent smooth beam with a surface current density. Brewer's model for the electron beam is simply a beam of electrons and does not postulate positive ion neutralization of the beam as some other common analyses do. The plasma, in this type of analysis, is treated as the limiting case of an electron beam in a plasma with zero charge density. Brewer, in his paper on the subject (7), obtains only a TM solution although a TE solution is not negated as in Trivelpiece's analysis. In this paper, an approximate TE solution is obtained, and interestingly, is shown to be coupled to the TM solution in such a manner that, if the TE solution is identically zero, then the TM solution is also zero, with the converse not necessarily true.

The third method of analysis is considered the most rigorous and is the most complex, mathematically. It consists of solving Maxwell's equations in an anisotropic media using Kales' (8) method of solution. This method provides an exact solution of the mathematical model as described above and further amplified in appendix A. Interesting features of this method are the requirement of coupled TM and TE modes of propagation which cannot be zero independently, and a mode degeneracy with non-orthogonal modes.

In this paper, the subscript o will be used to denote dc (static) quantities and the subscript 1, time varying quantities. Unless noted to the contrary, all quantities will be assumed to vary as

$$\bar{F} = \bar{F}_0(r, \theta, z) + \bar{F}_1(r, \theta, z)e^{j\omega t} \qquad (1.1)$$

with a z-dependence such that

$$\frac{\partial \bar{F}}{\partial z} = -\gamma \bar{F}_1(r, \theta, z) \qquad (1.2)$$

Following the usual procedure, the factor $e^{j\omega t}$ will be understood and not written explicitly.

2. Method I, Trivelpiece's "Slow Wave" Approximation

Following Trivelpiece (1), it is assumed that

$$\nabla \times \bar{E}_1 = -j\omega \bar{B}_1 \approx 0 \tag{2.1}$$

which then allows us to represent the electric field vector as derived from a scalar potential, i.e.

$$\bar{E}_1 = -\nabla \phi_1 \tag{2.2}$$

From Maxwell's equations, we have

$$\nabla \cdot \bar{D}_1 = \nabla \cdot (\underline{\underline{\epsilon}} \cdot E_1) = \nabla \cdot \underline{\underline{\epsilon}} \cdot \nabla \phi_1 = 0 \tag{2.3}$$

where $\underline{\underline{\epsilon}}$ is the tensor dielectric constant given by

$$\underline{\underline{\epsilon}} = \epsilon_0 \begin{Vmatrix} \epsilon_{11} & j\epsilon_{12} & 0 \\ -j\epsilon_{12} & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{Vmatrix} \tag{2.4}$$

A simple derivation of equation (2.4) is given as Appendix A although it is worth noting, in passing, that the same result for the case of a plasma alone may be obtained from the Boltzmann transport equation (10) with far fewer restrictions upon the derivation.

Proceeding formally, one then obtains

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\epsilon_{11}\frac{\partial \phi_1}{\partial r} + j\epsilon_{12}\frac{\partial \phi_1}{\partial \theta}\right] + \frac{1}{r}\frac{\partial}{\partial \theta}\left[-j\epsilon_{12}\frac{\partial \phi_1}{\partial r} + \epsilon_{11}\frac{1}{r}\frac{\partial \phi_1}{\partial r}\right]$$

$$+ \frac{\partial}{\partial z}\left[\epsilon_{33}\frac{\partial \phi_1}{\partial z}\right] = 0 \tag{2.5}$$

If a product type of solution is assumed,

$$\phi_1 = R(r)\,\Theta(\theta)\,e^{-\gamma z} \tag{2.6}$$

the following differential equation is then obtained

$$\frac{\partial^2 \phi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi_1}{\partial \theta^2} + \frac{\epsilon_{33}}{\epsilon_{11}}\gamma^2 \phi_1 = 0 \tag{2.7}$$

the solution of which is

$$\phi_1 = \left[A J_m(Tr) + B N_m(Tr)\right]\left[C\cos(m\theta) + D\sin(m\theta)\right] e^{-\gamma z} \tag{2.8}$$

where A, B, C, and D are arbitrary constants. Following the usual procedure in waveguide type propagation problems, this will be expressed as

$$\phi_1 = \left[A J_m(Tr) + B N_m(Tr)\right] e^{-jm\theta - \gamma z} \tag{2.9}$$

where

$$T^2 = \gamma^2 \frac{\epsilon_{33}}{\epsilon_{11}} \tag{2.10}$$

We then have

$$E_{1r} = -\left[A T J_m'(Tr) + B T N_m'(Tr)\right] e^{-jm\theta - \gamma z} \tag{2.11}$$

$$E_{1\theta} = \left[A \frac{jm}{r} J_m(Tr) + B \frac{jm}{r} N_m(Tr)\right] e^{-jm\theta - \gamma z} \tag{2.12}$$

$$E_{1z} = \left[A\gamma J_m(Tr) + B\gamma N_m(Tr)\right] e^{-jm\theta-\gamma z} \tag{2.13}$$

where the prime indicates differentiation with respect to the total argument.

While it is implicit in the assumption behind equation (2.2) that the time varying magnetic field is essentially zero with respect to the electric field, the curl H equation

$$\nabla \times \bar{H}_1 = j\omega \underline{\underline{\epsilon}} \cdot E_1 \tag{2.14}$$

will be used to obtain approximate values of the magnetic field components. H_z will be set equal to zero, otherwise the existence of another mode of propagation would be allowed which would contradict the basic assumption, equation (2.2). Taking the components of equation (2.14), one obtains

$$\frac{1}{r}\frac{\partial H_{1z}}{\partial \theta} - \frac{\partial H_{1\theta}}{\partial z} = j\omega\epsilon_0\left[\epsilon_{11}E_{1r} + j\epsilon_{12}E_{1\theta}\right] \tag{2.15}$$

$$\frac{\partial H_{1r}}{\partial z} - \frac{\partial H_{1z}}{\partial r} = j\omega\epsilon_0\left[-j\epsilon_{12}E_{1r} + \epsilon_{11}E_{1\theta}\right] \tag{2.16}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rH_{1\theta}\right) - \frac{\partial H_{1z}}{\partial r} = j\omega\epsilon_0\epsilon_{33}E_{1z} \tag{2.17}$$

We then obtain (where the primes again denote differentiation with respect to the total argument)

$$B_{1r} = -\frac{j\omega\mu\epsilon_0}{\gamma}\Big[j\epsilon_{12}\big(ATJ_m'(Tr) + BTN_m'(Tr)\big) + \tag{2.18}$$

$$\epsilon_{33}\frac{jm}{r}\big(AJ_m(Tr) + BN_m(Tr)\big)\Big]$$

$$B_{1\theta} = \frac{j\omega\mu\epsilon_0}{\gamma}\Big[-\epsilon_{11}\big(ATJ_m'(Tr) + BTN_m'(Tr)\big) +$$

$$\frac{\epsilon_{12}jm}{r}\big(AJ_m(Tr) + BN_m(Tr)\big)\Big] \tag{2.19}$$

From this point on, only the axially symmetric case (n=0) will be considered. Also, only the time varying field quantities are involved so the subscript 1 will be omitted to simplify the notation.

$$E_z = A\gamma J_0(Tr) + B\gamma N_0(Tr) \tag{2.20}$$

$$E_r = ATJ_1(Tr) + BTN_1(Tr) \tag{2.21}$$

$$H_\theta = \frac{j\omega\epsilon_0}{\gamma}\Big[\epsilon_{11}\big(ATJ_1(Tr) + BTN_1(Tr)\big)\Big] \tag{2.22}$$

$$H_{1r} = -\frac{j\omega\epsilon_0}{\gamma}\left[-j\epsilon_{12}\left(ATJ_1(Tr) + BTN_1(Tr)\right)\right] \tag{2.23}$$

Equations (20) through (23) are the field quantities derived from equations (1) and (14), to be used in the boundary matching problem.

As was mentioned in the introduction, it is impossible to match a TM solution alone to the "sheath helix" using the standard boundary conditions as given by Pierce (4) or Beck (9). Some form of the TE solution must also be used. For this analysis, a free space TE solution will be assumed, with more to be said upon this assumption later. It is to be noted that, up to this point, no assumptions, other than those made by Trivelpiece (1) have been made. For a detailed justification of these assumptions, the reader is referred to Trivelpiece's work. Extracting, the appropriate TE solution from Beck (9), the boundary value problem then becomes (referring to Fig. 1)

<u>Region I</u> <u>$0 \le r \le a$</u> (Electron beam within a plasma)

From Appendix A,

$$\underline{\underline{\epsilon}} = \epsilon_0 \begin{Vmatrix} \epsilon_1 & j\epsilon_2 & 0 \\ -j\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{Vmatrix} \tag{2.24}$$

$$E_z = A_1 \gamma J_0(T_1 r) \qquad H_z = A_2 I_0(pr)$$

$$H_\theta = j\frac{\omega \epsilon_0 \epsilon_1}{\gamma} A_1 T_1 J_1(T_1 r) \qquad E_\theta = -j\frac{\omega\mu}{p} A_2 I_1(pr) \tag{2.25}$$

<u>Region</u> <u>II</u> <u>a ≤ r ≤ b</u> (Plasma region)

From Appendix A

$$\underline{\underline{\epsilon}} = \epsilon_0 \begin{Vmatrix} \epsilon_{11} & j\epsilon_{12} & 0 \\ -j\epsilon_{12} & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{Vmatrix} \tag{2.26}$$

$$E_z = B_1 \gamma J_0(T_2 r) + B_2 \gamma N_0(T_2 r)$$

$$H_\theta = j\frac{\omega \epsilon_0 \epsilon_{11}}{\gamma} \left[B_1 T_2 J_1(T_2 r) + B_2 T_2 N_1(T_2 r) \right]$$

$$H_z = A_2 I_0(pr) \tag{2.27}$$

$$E_\theta = -\frac{j\omega\mu}{p} A_2 I_1(pr)$$

<u>Region</u> <u>III</u> <u>b ≤ r ≤ c</u> (Free space)

$$\begin{aligned} E_z &= C_1 I_0(pr) + C_2 K_0(pr) \\ H_\theta &= j\frac{\omega\epsilon_0}{p} C_1 I_1(pr) - j\frac{\omega\epsilon_0}{p} C_2 K_1(pr) \\ H_z &= C_3 I_0(pr) + C_4 K_0(pr) \\ E_\theta &= -j\frac{\omega\mu}{p} C_3 I_1(pr) + j\frac{\omega\mu}{p} C_4 K_1(pr) \end{aligned} \tag{2.28}$$

Region IV $r \geq c$ (Free space)

$$
\begin{aligned}
E_z &= D_1 K_0(pr) \\
H_\theta &= -\frac{j\omega\epsilon_0}{p} D_1 K_1(pr) \\
H_z &= D_2 K_0(pr) \\
E_\theta &= \frac{j\omega\mu}{p} K_1(pr)
\end{aligned}
\tag{2.29}
$$

The boundary conditions at the helix are:

$$
\begin{aligned}
E_z^i &= E_z^e \\
E_\theta^i &= E_\theta^e \\
E_z^{i,e} &= -E_\theta^{i,e} \cot\psi \\
H_z^i + H_\theta^i \cot\psi &= H_z^e + H_\theta^e \cot\psi
\end{aligned}
\tag{2.30}
$$

As the general problem is rather complex, several simpler cases will be considered first. These are:

Case I, the case of an electron beam completely filling the interior of the plasma and the helix, i.e., a region I and IV problem.

Case II, the case of an electron beam of radius a, less than the diameter of the helix, with the plasma filling the helix, i.e., a region I, II, and IV problem.

Case III, the case of an electron beam of radius b, passing through a plasma of radius b, surrounded by a free space region and a helix at radius c, with free space outside the helix, i.e., a region I, III and IV problem.

Case IV, the case of an electron beam of radius a passing through a plasma

of radius b surrounded by free space and a helix at radius c with free space outside the helix, i.e., a region I, II, III, and IV problem.

Determinantal relationships are obtained for all four of these cases with the algebra relegated to Appendix B. The determinantal relationships are given below.

Case I

$$\frac{-1}{j\gamma c} + \frac{\cot^2\psi\, k^2 I_1(pc)}{p^2}\left[\frac{K_1(pc)}{K_0(pc)} + \frac{j\gamma\, \epsilon_3 \epsilon_1\, J_1(T_1 c)}{J_0(T_1 c)}\right] = 0 \tag{2.31}$$

CASE II

$$\left[\frac{-jJ_0(T_2b)}{bI_1(\rho b)K_1(\rho b)} + \frac{\cot^2\psi\, k^2}{\gamma}\left(-j\sqrt{\epsilon_{11}\epsilon_{33}}\, J_1(T_2b) + \frac{K_1(\rho b)}{K_0(\rho b)}J_0(T_2b)\right)\right]\left[\frac{N_0(T_2a)}{J_0(T_1a)} - \sqrt{\frac{\epsilon_{11}\epsilon_{33}}{\epsilon_1\epsilon_3}}\frac{N_1(T_2a)}{J_1(T_1a)}\right] -$$

$$\left[\frac{-jN_0(T_2b)}{bI_1(\rho b)K_1(\rho b)} + \frac{\cot^2\psi\, k^2}{\gamma}\left(-j\sqrt{\epsilon_{11}\epsilon_{33}}\, N_1(T_2b) + \frac{K_1(\rho b)}{K_0(\rho b)}N_0(T_2b)\right)\right]\left[\frac{J_0(T_2a)}{J_0(T_1a)} - \sqrt{\frac{\epsilon_{11}\epsilon_{33}}{\epsilon_1\epsilon_3}}\frac{J_1(T_2a)}{J_1(T_1a)}\right] = 0 \qquad (2.32)$$

CASE III

$$\left[\frac{K_0(\rho c)}{I_1(\rho c)K_1(\rho c)}\right]\left[\frac{I_0(\rho b)}{J_0(T_1b)} - \frac{jI_1(\rho b)}{\sqrt{\epsilon_1\epsilon_3}\,J_1(T_1b)}\right] -$$

$$\left[\frac{I_0(\rho c)}{I_1(\rho c)K_1(\rho c)} - \frac{k^2\cot^2\psi}{\rho^2\, K_0(\rho c)}\right]\left[\frac{K_0(\rho c)}{J_0(T_1b)} + \frac{jK_1(\rho b)}{\sqrt{\epsilon_1\epsilon_3}\,J_1(T_1b)}\right] = 0 \qquad (2.33)$$

CASE IV

$$\left[-j\sqrt{\epsilon_{33}\,\epsilon_{11}}\,K_0(\rho b)\Big(N_1(T_2 b)G - J_1(T_2 b)H\Big) + K_1(\rho b)\Big(N_0(T_2 b)G - J_0(T_2 b)H\Big)\right] \times$$

$$\left[\frac{I_0(\rho c)}{I_1(\rho c)} - \frac{k^2 \cot^2\psi}{\rho^2}\,\frac{K_1(\rho c)}{K_0(\rho c)}\right] - \left[\frac{K_0(\rho c)}{I_1(\rho c)}\right] \times \tag{2.34}$$

$$\left[-j\sqrt{\epsilon_{33}\,\epsilon_{11}}\,I_0(\rho b)\Big(N_1(T_2 b)G - J_1(T_2 b)H\Big) - I_1(\rho b)\Big(N_0(T_2 b)G - J_0(T_2 b)H\Big)\right] = 0$$

WHERE

$$G = J_0(T_2 a)J_1(T_1 a) - \sqrt{\frac{\epsilon_{11}\,\epsilon_{33}}{\epsilon_1\,\epsilon_3}}\,J_1(T_2 a)\,J_0(T_1 a) \tag{2.35}$$

$$H = N_0(T_2 a)J_1(T_1 a) - \sqrt{\frac{\epsilon_{11}\,\epsilon_{33}}{\epsilon_1\,\epsilon_3}}\,N_1(T_2 a)\,J_0(T_1 a) \tag{2.36}$$

3. Method II, (Extension Of Brewer's Method)

This method is an extension of the method of Rigrod and Lewis (6) and Brewer (7). Attention is also invited to Beck's (9) excellent treatment from which this presentation proceeds.

Let us consider an electron beam with no angular variation in charge density, fields or electron motion, i.e.,

$$\frac{\partial}{\partial \theta} = 0$$

We may then write the Lorentz force equation in cylindrical coordinates as

$$\ddot{r} - r\dot{\theta}^2 = -\eta\left[E_r + B_z r\dot{\theta}\right] \tag{3.1}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -\eta\left[-B_z\dot{r} + B_r\dot{z}\right] \tag{3.2}$$

$$\ddot{z} = -\eta\left[E_z - B_z r\dot{\theta}\right] \tag{3.3}$$

provided we assume as does Beck and Brewer, that the contribution of $E_{1\theta}$ to the right side of (3.2) is small with respect to $B_z\dot{r}$. More will be said of this approximation later. Using Busch's theorem from electron optics, we may write

$$r^2\dot{\theta} = \frac{\eta}{2}\left[B_z r^2 - r_c^2 B_0\right] \tag{3.4}$$

Where r_c is the initial radius of each electron and B_0 the z-directed

magnetic field at that point (assumed constant). This may be stated as

$$\dot{\theta} = \frac{1}{2}\left[\omega_c - \omega_0 \frac{r_c^2}{r^2}\right] \tag{3.5}$$

Substituting this into (3.1), one obtains

$$\ddot{r} = -\eta E_r + \frac{r}{2}\left[\omega_0^2 \frac{r_c^4}{r^4} - \omega_c^2\right] \tag{3.6}$$

Let us now perturb (3.5) by letting

$$\dot{\theta} = \dot{\theta}_0 + \dot{\theta}_1 \qquad\qquad r = r_0 + r_1$$

to obtain

$$\dot{\theta}_1 = \frac{r_1}{r_0} \cdot \frac{\omega_0 r_c^2}{r^2} = \frac{r_1}{r_0}\Omega^2 \tag{3.7}$$

Next, assuming that the DC ripple or scallop on the beam is small, let

$$\ddot{r} = -\eta E_{r_0} + \frac{r}{2}\left[\omega_0^2 \frac{r_c^4}{r^4} - \omega_c^2\right] \approx 0 \tag{3.8}$$

Perturbing this expression as before yields

$$r_1 = \frac{\eta E_{r_1}}{(\omega + j\gamma u_0)^2 - \Omega^2} \tag{3.9}$$

Perturbing (3.2) and disregarding the effects of AC magnetic fields on electron motion compared with electric field effects

$$z_1 = \frac{\eta E_z}{(\omega + j\gamma u_0)^2} \tag{3.10}$$

Applying the continuity equation

$$\nabla \cdot \bar{J} = -\frac{\partial \rho}{\partial t} \tag{3.11}$$

to the AC charge density, one obtains

$$\rho_1 = \frac{j\rho_0 \nabla \cdot \bar{v}_1}{(\omega + j\gamma u_0)} = \frac{j\rho_0}{(\omega + j\gamma u_0)} \left[\frac{1}{r} \frac{\partial}{\partial r}(r v_{1r}) + \frac{\partial v_{1z}}{\partial z} \right] \tag{3.12}$$

At this point, let us examine what has been done. The equilibrium relation (3.8) has been perturbed to obtain (3.9). As stated in the introduction, electrons are assumed to interact only through the electric field and the above relations are derived as if the individual electrons compose a stream with continuity of charge in this stream maintained through equation (3.11). If a second group of charged particles were present, such that the DC equilibrium condition remained valid, comparable expressions for the time varying quantities could also be written for the second group of charged particles. The model of the plasma set forth in the introduction fits these conditions quite well since overall DC neutrality from the plasma ions and electrons is maintained. Assuming that the continuity equation holds (which it must as no mechanism for loss of charged particles has been assumed in the model), one can write expressions comparable to (3.9), (3.10), and (3.11) (with $U_o = 0$) for both ions and electrons, but, since the mass of the ions is so much greater than that of the electrons, their effects are extremely small at any frequency considerably above one megacycle per second and will be ignored. Designating the beam electrons

by subscript $\underline{b}$ and the plasma electrons by subscript $\underline{a}$ and dropping the subscript $\underline{1}$ from field quantities since only ac fields are to be treated, the following relations are obtained.

$$\rho_{1b} = \frac{j\rho_{ob}}{(\omega + j\gamma u_0)}\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_{1rb}) + \frac{\partial v_{1zb}}{\partial z}\right] \tag{3.13}$$

$$\rho_{1a} = \frac{j\rho_{oa}}{\omega}\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_{1ra}) + \frac{\partial v_{1za}}{\partial z}\right] \tag{3.14}$$

$$r_{1b} = \frac{\eta E_r}{(\omega + j\gamma u_0)^2 - \Omega_b^2} \tag{3.15}$$

$$r_{1a} = \frac{\eta E_r}{\omega^2 - \Omega_a^2} \tag{3.16}$$

$$z_{1b} = \frac{\eta E_z}{(\omega + j\gamma u_0)^2} \tag{3.17}$$

$$z_{1a} = \frac{\eta E_z}{\omega^2} \tag{3.18}$$

$$v_{1rb} = \frac{j(\omega + j\gamma u_0)\eta E_r}{(\omega + j\gamma u_0)^2 - \Omega_b^2} \tag{3.19}$$

$$v_{1ra} = \frac{j\omega\eta E_r}{\omega^2 - \Omega_a^2} \tag{3.20}$$

Assuming that

$$J_{1r} = \rho_{oa} v_{1ra} + \rho_{ob} v_{1rb} \tag{3.21}$$

and, from Maxwell's equations in a media containing charge

$$E_r = \frac{\gamma}{p^2}\frac{\partial E_z}{\partial r} - \frac{j\omega\mu}{p^2} J_{1r} \tag{3.22}$$

From (3.13), (3.14), (3.19) and (3.20)

$$J_{1r} = \frac{\rho_{oa} j\omega\eta E_r}{\omega^2 - \Omega_a^2} + \frac{\rho_{ob} j(\omega + j\gamma u_o)\eta E_r}{(\omega + j\gamma u_o)^2 - \Omega_b^2} \tag{3.23}$$

It will be convenient to write

$$J_{1r} = F_1 E_r \tag{3.24}$$

From (3.22) and (3.23)

$$E_r = \frac{\gamma}{p^2}\frac{\partial E_z}{\partial r} - \frac{j\omega\mu}{p^2} F_1 E_r \tag{3.25}$$

At this point, Beck shows that the second term of the last relationship is small with respect to the first for electron beams and may be ignored. This approximation is not as clearly well taken in this analysis and will not be made. Comment on the effect of making it will be made later.

Solving for E_r

$$E_r = \frac{\gamma}{p^2 + j\omega\mu F_1} \frac{\partial E_z}{\partial r} \tag{3.26}$$

Small DC beam scalloping has already been assumed and since the assumption that the magnitude of the perturbation is small is inherent in the perturbational approach, $r_c \approx r$. Further, if we assume that the dc magnetic field strength is everywhere constant and the cathode is not shielded, $B_o \approx B_z$ and

$$\Omega^2 \approx \omega_c^2 \tag{3.27}$$

Making this approximation and writing

$$J_{1z} = \rho_{oa} v_{1za} + \rho_{ob} v_{1zb} + \rho_{1b} u_0 \tag{3.28}$$

one obtains

$$J_{1z} = E_z \left[\frac{j\eta\rho_{oa}}{\omega} + \frac{j\eta\rho_{ob}}{\omega + j\gamma u_0} + \frac{\gamma\rho_{ob}\eta u_0}{(\omega + j\gamma u_0)^2} \right] - \frac{\gamma\rho_{ob}\eta u_0}{[(\omega + j\gamma u_0)^2 - \omega_c^2][p^2 + j\omega\mu F_1]} \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial E_z}{\partial r} \right) \tag{3.29}$$

Writing this as

$$J_{1z} = L E_z - F_3 \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial E_z}{\partial r} \right) \tag{3.30}$$

and

$$J_{1r} = F_1 E_r = \frac{\gamma F_1}{(p^2 + j\omega\mu F_1)} \frac{\partial E_z}{\partial r} \tag{3.31}$$

and from Maxwell's equations in a media containing charge

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial E_z}{\partial r}\right) - p^2 E_z = \frac{p^2}{j\omega\epsilon_0} J_{1z} - \frac{j\gamma}{\omega\epsilon_0}\frac{1}{r}\frac{\partial}{\partial r}\left(r J_{1r}\right) \tag{3.32}$$

there results

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial E_z}{\partial r}\right) + \gamma_r^2 E_z = 0 \tag{3.33}$$

where

$$\gamma_r^2 = - \frac{p^2 (j\omega\epsilon_0 + L)(p^2 + j\omega\mu F_1)}{(p^2 + j\omega\mu F_1)(j\omega\epsilon_0 + p^2 F_3) - \gamma^2 F_1} \tag{3.34}$$

The solution of this equation is,

$$E_z = C_1 J_0(\gamma_r r) + C_2 N_0(\gamma_r r) \tag{3.35}$$

The notation at this point has become rather cumbersome. For clarity of presentation and ease of manipulation, some arbitrarily defined constants must be used. Although the use of the components of the dielectric tensors, as derived in Appendix A, is objectionable in that an equivalent dielectric is not being used in this analysis, these quantities are familiar to workers in the field and their use will facilitate comparison of this method with the other two methods presented in this paper. Introducing these expressions and a quantity

$$P_h^2 = p^2 + j\omega\mu F_1 \tag{3.36}$$

the significance of which will appear later, we have

$$F_1 = j\omega\epsilon_0(\epsilon_1 - 1) + \gamma u_0(\epsilon_{11} - \epsilon_1)\epsilon_0 \tag{3.37}$$

$$L = j\omega\epsilon_0(\epsilon_3 - 1) \tag{3.38}$$

$$F_3 = \frac{\gamma u_0 \epsilon_0(\epsilon_1 - \epsilon_{11})}{P_h^2} \tag{3.39}$$

$$\gamma_r^2 = -\frac{j\omega\epsilon_0\epsilon_3 p^2 P_h^2}{j\omega\epsilon_0 P_h^2 + p^2\gamma u_0(\epsilon_1 - \epsilon_{11}) - \gamma^2 F_1} \tag{3.40}$$

At this point, a convenient check on the development exists. If we let ρ_{ob} go to zero and make all the substitutions indicated in (3.34), we obtain

$$\gamma_{rp}^2 = \frac{\left[1 - \frac{\omega_p^2}{\omega^2}\right]\left[-p^2 - \frac{k^2\omega_p^2}{\omega^2 - \omega_c^2}\right]}{1 + \frac{\omega_p^2}{\omega_c^2 - \omega^2}} \tag{3.41}$$

The second term in the second bracket in the numerator can be shown to be absent if, in equation (3.22), the term containing J_{1r} is ignored as

negligible. If this term is omitted, the result is then identical to that obtained by Method I (Trivelpiece's "slow wave" approximation). It is obvious that this term is not always negligible, even for very slow waves ($|\gamma^2| >> k^2$). Similarly, if the term containing J_{1r} is ignored in (3.22) and if we let $\rho_{oa} = 0$, the results can be shown to agree with Beck's (9) confined beam development.

Thus far, the TM solution has been obtained with the principal approximation being that $E_{1\theta}$ is negligible with respect to $B_z\dot{r} - B_r\dot{z}$ in (3.2). This, in effect states that the coupling of the TE mode, from which $E_{1\theta}$ is obtained, to the TM mode is negligible. An approximate TE solution which does not ignore the coupling of the TM to the TE mode will now be derived. From Maxwell's equations in a region containing charge, we obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_z}{\partial r}\right) + (k^2+\gamma^2)H_z = -\frac{1}{r}\frac{\partial}{\partial r}(rJ_\theta) \tag{3.42}$$

$$H_r = \frac{-\gamma}{\gamma^2+k^2}\left[\frac{\partial H_z}{\partial r} + J_\theta\right] \tag{3.43}$$

$$E_\theta = \frac{-j\omega\mu}{\gamma}H_r \tag{3.44}$$

and, from Appendix 1

$$v_{1\theta a} = -\eta \left[\frac{-j\omega E_\theta - \omega_c E_r}{\omega^2 - \omega_c^2} \right] \quad (3.45)$$

$$v_{1\theta b} = -\eta \left[\frac{-j(\omega + j\gamma u_0) E_\theta - \omega_c E_r}{(\omega + j\gamma u_0)^2 - \omega_c^2} \right] \quad (3.46)$$

Assuming

$$J_\theta = \rho_{0a} v_{1\theta a} + \rho_{0b} v_{1\theta b} \quad (3.47)$$

we obtain

$$J_\theta = E_\theta \left[\frac{j\omega\omega_p^2 \epsilon_0}{\omega_c^2 - \omega^2} - \frac{j(\omega + j\gamma u_0)\omega_{pb}^2 \epsilon_0}{(\omega + j\gamma u_0)^2 - \omega_c^2} \right] + \quad (3.48)$$

$$E_r \left[\frac{\omega_c \epsilon_0 \omega_p^2}{\omega_c^2 - \omega^2} - \frac{\omega_c \epsilon_0 \omega_{pb}^2}{(\omega + j\gamma u_0)^2 - \omega_c^2} \right]$$

which, for brevity, will be written

$$J_\theta = F_1 E_\theta + \omega_c \epsilon_0 (\epsilon_1 - 1) E_r \quad (3.49)$$

Substituting (3.44), (3.26), (3.36), into (3.49), we obtain

$$J_\theta = \left[\frac{p^2}{p_h^2} - 1 \right] \frac{\partial H_z}{\partial r} + \frac{\gamma p^2 \omega_c \epsilon_0 (\epsilon_1 - 1)}{p_h^4} \frac{\partial E_z}{\partial r} \quad (3.50)$$

We now have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_z}{\partial r}\right) - p^2 H_z = \left[1 - \frac{p^2}{p_h^2}\right]\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_z}{\partial r}\right)$$

$$- \frac{\gamma p^2 \omega_c \epsilon_0(\epsilon_1 - 1)}{p_h^4}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) \tag{3.51}$$

where

$$E_z = A_1 J_0(\gamma_r r) + A_2 N_0(\gamma_r r) \tag{3.52}$$

Equivalently, let us write

$$\frac{p^2}{p_h^2}H_z'' + \frac{p^2}{p_h^2}\frac{H_z'}{r} - p^2 H_z = -\frac{\gamma p^2 \omega_c \epsilon_0(\epsilon_1 - 1)}{p_h^4}\left[E_z'' + \frac{E_z'}{r}\right] \tag{3.53}$$

where the primes denote differentiation with respect to r.

A solution to this inhomogeneous Bessel equation may be obtained by letting

$$H_z = A\,u_1(r) + B\,u_2(r) \tag{3.54}$$

$$E_z = C\,u_1(r) \tag{3.55}$$

Making these substitutions in (3.53)

$$U_1'' + \frac{U_1'}{r} + U_1\left[\frac{-AP_h^4}{AP_h^2 + C\gamma\omega_c\epsilon_0(\epsilon_1 - 1)}\right] +$$

$$\left[\frac{BP_h^2}{AP_h^2 + C\gamma\omega_c\epsilon_0(\epsilon_1 - 1)}\right]\left[U_2'' + \frac{U_2'}{r} - P_h^2 U_2\right] = 0 \qquad (3.56)$$

For brevity, this will be written

$$f(U_1) + f(U_2) = 0 \qquad (3.57)$$

If we require that

$$\frac{-p^2 A}{A\frac{p^2}{P_h^2} + C\frac{\gamma p^2 \omega_c\epsilon_0(\epsilon_1 - 1)}{P_h^4}} = \gamma_r^2 \qquad (3.58)$$

as we must to be consistent with (3.55), $f(U_1)$ will then be identically zero. We may, without loss of generality, set B=C=1. We then have

$$U_2'' + \frac{U_2'}{r} - P_h^2 U_2 = 0 \qquad (3.59)$$

the solution of which is

$$U_2 = C_1 I_0(P_h r) + C_2 K_0(P_h r) \qquad (3.60)$$

which then furnishes the reason for the somewhat peculiar choice of constants made in (3.36) and

$$A = F_2 = -\frac{\gamma_r^2\gamma\omega_c\epsilon_0(\epsilon_1 - 1)}{P_h^2(P_h^2 + \gamma_r^2)} \qquad (3.61)$$

The complete solution is then

$$H_z = C_1 F_2 J_0(\gamma_r r) + C_2 F_2 N_0(\gamma_r r) + C_3 I_0(\beta_r r) + C_4 K_0(\beta_r r) \tag{3.62}$$

As was stated earlier, the plasma alone is considered as the limiting case of zero electron beam charge density. Hence we may write expressions for the plasma region alone which are identical in form to the preceding with $\omega_{pb}^2 = 0$. The additional subscript p will be added to distinguish between the two regions.

As stated in the introduction, a distinguishing feature of this method of analysis is the replacement of the rippled beam with an equivalent cylindrical beam with a surface current density. Expressions for the surface current density will now be developed. Following Beck's procedure, we write (at r=d)

$$G_z = \rho_{ob} r_{1b} u_0 \tag{3.63}$$

$$G_\theta = \rho_{oa} r_{1a} \frac{\omega_c d}{2} + \rho_{ob} r_{1b} \frac{\omega_c d}{2} \tag{3.64}$$

Using (3.15) and (3.16), we obtain

$$G_z = \epsilon_0 u_0 (\epsilon_1 - \epsilon_{11}) E_r(d) \tag{3.65}$$

$$G_\theta = \frac{\omega_c d \epsilon_0}{2}(\epsilon_1 - 1) E_r(d) \tag{3.66}$$

From Maxwell's equations, we obtain

$$H_\theta = \frac{j\omega\epsilon_0}{\gamma} E_r + \frac{1}{\gamma} J_r \tag{3.67}$$

$$E_\theta = \frac{-j\omega\mu}{p^2}\left[\frac{\partial H_z}{\partial r} + J_\theta\right] \tag{3.68}$$

Using (3.50), (3.24), (3.26) and manipulating the various constants

$$H_\theta = \frac{1}{j\omega\mu}\left[1 + \frac{\gamma^2}{P_h^2}\right]\frac{\partial E_z}{\partial r} \tag{3.69}$$

$$E_\theta = \frac{-j\omega\mu}{P_h^2}\left[\frac{\partial H_z}{\partial r} + \frac{\gamma\omega_c\epsilon_0(\epsilon_1 - 1)}{P_h^2}\frac{\partial E_z}{\partial r}\right] \tag{3.70}$$

$$H_\theta = -\frac{\gamma_r}{j\omega\mu}\left[1 + \frac{\gamma^2}{P_h^2}\right]\left[A_1 J_1(\gamma_r r) + A_2 N_1(\gamma_r r)\right] \tag{3.71}$$

$$E_\theta = -\frac{j\omega\mu}{p_h}\left[C_3 I_1(p_h r) - C_4 K_1(p_h r)\right] +$$

$$\frac{j\omega\mu}{p_h^2}\left[\frac{\gamma\omega_c\epsilon_0(\epsilon_1 - 1)}{p_h^2 + \gamma_r^2}\right]\gamma_r\left[A_1 J_1(\gamma_r r) + A_2 N_1(\gamma_r r)\right] \tag{3.72}$$

All the field quantities required for the boundary value problem have now been obtained. Now we shall establish the boundary conditions. From Stratton (11), we obtain the requirements that tangential E must be continuous and that

$$\hat{n} \times \left[\bar{H}_{II} - \bar{H}_{I}\right] = \bar{G} \tag{3.73}$$

which then provides the following set of boundary conditions

$$\begin{aligned} E_{\theta_I} &= E_{\theta_{II}} & \qquad E_{z_I} &= E_{z_{II}} \\ H_{\theta_{II}} &= H_{\theta_I} + G_z & \qquad H_{z_{II}} &= H_{z_I} - G_\theta \end{aligned} \tag{3.74}$$

The boundary value problem is then:

<u>Region I</u> (beam and Plasma)

$$E_z = A_1 J_0(\gamma_r r) \tag{3.75}$$

$$H_\theta = -\frac{\gamma_r}{j\omega\mu}\left[1 + \frac{\gamma^2}{p_h^2}\right] A_1 J_1(\gamma_r r) \tag{3.76}$$

$$H_z = A_1 F_2 J_0(\gamma_r r) + A_3 I_0(p_h r) \tag{3.77}$$

$$E_\theta = \frac{j\omega\mu}{p_h^2} \frac{\gamma \gamma_r \omega_c \epsilon_0 (\epsilon_1 - 1)}{(p_h^2 + \gamma_r^2)} A_1 J_1(\gamma_r r) - \frac{j\omega\mu}{p_h} A_3 I_1(p_h r) \tag{3.78}$$

<u>Region</u> <u>II</u> (Plasma)

$$E_z = B_1 J_0(\gamma_{rp} r) + B_2 N_0(\gamma_{rp} r) \tag{3.79}$$

$$H_\theta = -\frac{\gamma_{rp}}{j\omega\mu}\left[1 + \frac{\gamma^2}{p_{hp}^2}\right]\left[B_1 J_1(\gamma_{rp} r) + B_2 N_1(\gamma_{rp} r)\right] \tag{3.80}$$

$$H_z = B_1 F_{2p} J_0(\gamma_{rp} r) + B_2 F_{2p} N_0(\gamma_{rp} r) + B_3 I_0(p_{hp} r) + B_4 K_0(p_{hp} r) \tag{3.81}$$

$$E_\theta = \frac{j\omega\mu \gamma \gamma_{rp} \omega_c \epsilon_0 (\epsilon_{11} - 1)}{p_{hp}^2 (p_{hp}^2 + \gamma_{rp}^2)}\left[B_1 J_1(\gamma_{rp} r) + B_2 N_1(\gamma_{rp} r)\right]$$

$$- \frac{j\omega\mu}{p_{hp}}\left[B_3 I_1(p_{hp} r) - B_4 K_1(p_{hp} r)\right] \tag{3.82}$$

<u>Region</u> <u>III</u> (Free space)

$$E_z = C_1 I_0(pr) + C_2 K_0(pr) \tag{3.83}$$

$$H_\theta = \frac{j\omega\epsilon_0}{p}\left[C_1 I_1(pr) - C_2 K_1(pr)\right] \tag{3.84}$$

$$H_z = C_3 I_0(pr) + C_4 K_0(pr) \tag{3.85}$$

$$E_\theta = -\frac{j\omega\mu}{p}\left[C_3 I_1(pr) - C_4 K_1(pr)\right] \tag{3.86}$$

<u>Region</u> <u>IV</u> (Free Space)

$$E_z = D_1 K_0(pr) \tag{3.87}$$

$$H_\theta = -\frac{j\omega\epsilon_0}{p} D_1 K_1(pr) \tag{3.88}$$

$$H_z = D_2 K_0(pr) \tag{3.89}$$

$$E_\theta = \frac{j\omega\mu}{p} D_2 K_1(pr) \tag{3.90}$$

As in Section two, the same four cases will be considered with the details contained in Appendix C. It must be noted that, since this method calls

for a rippled beam, space for the ripple must be allowed. Cases one and two must then be considered as limiting situations where the free space region has shrunk to zero.

4. Method III, Solution of Field Equations by Kales' Method.

In Appendix A, tensor dielectric constants have been derived which take into account the effects of the electronic motion in an electron beam and plasma. Using the tensor dielectric constant, we may then solve Maxwell's equations

$$\nabla \times \bar{E} = -j\omega\mu\bar{H} \tag{4.1}$$

$$\nabla \times \bar{H} = \bar{J} + j\omega\bar{D} \tag{4.2}$$

$$\nabla \cdot \bar{B} = 0 \tag{4.3}$$

$$\nabla \cdot \bar{D} = \rho \tag{4.4}$$

for a region with zero charge and current density, since the effects of electronic motion are already taken into account by the tensor dielectric constant. It will be convenient to change the notation slightly so that

$$\underline{\underline{\epsilon}} = \begin{Vmatrix} \epsilon_{11} & j\epsilon_{12} & 0 \\ -j\epsilon_{12} & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{Vmatrix} \qquad \bar{D} = \underline{\underline{\epsilon}} \cdot \bar{E} \tag{4.5}$$

the net effect of which is to include the factor ϵ_0 in each of the ϵ_{jk}

Kales (8) developed a method for the solution of Maxwell's equations in anisotropic media such that the dielectric constant was isotropic but the permeability, μ, was a tensor of the same form as (4.5). This method has been used by Stafford (13) to study resonance phenomena in a plasma column, and by Johnson (14) for the case of a plasma in a cylindrical drift tube. In the presentation that follows, Stafford's notation will be generally followed.

Proceeding, we may write

$$\nabla \times \bar{E} = \nabla_t \times \bar{E} + \hat{a}_z \times (-\gamma)\bar{E} \tag{4.6}$$

$$\nabla_t \times \bar{E} = \nabla_t \times \bar{E}_t + \nabla_t E_z \times \hat{a}_z \tag{4.7}$$

$$\nabla \times \bar{H} = \nabla_t \times \bar{H} + \hat{a}_z \times (-\gamma)\bar{H} \tag{4.8}$$

$$\nabla_t \times \bar{H} = \nabla_t \times \bar{H}_t + \nabla_t H_z \times \hat{a}_z \tag{4.9}$$

Separating transverse and longitudinal components

$$\nabla_t \times \bar{E}_t = -j\omega\mu H_z \hat{a}_z \tag{4.10}$$

$$\hat{a}_z \times (-\gamma \bar{E}_t - \nabla_t E_z) = -j\omega\mu \bar{H}_t \tag{4.11}$$

$$\nabla_t \times \bar{H}_t = j\omega \epsilon_{33} E_z \hat{a}_z \tag{4.12}$$

$$\hat{a}_z \times (-\gamma \bar{H}_t - \nabla_t H_z) = j\omega\epsilon_{11} \bar{E}_t + \omega\epsilon_{12} \hat{a}_z \times \bar{E}_t \tag{4.13}$$

solving (4.11) for $\hat{a}_z \times \bar{E}_t$

$$\hat{a}_z \times \bar{E}_t = \frac{j\omega\mu}{\gamma} \bar{H}_t + \hat{a}_z \times \frac{\nabla_t E_z}{-\gamma} \tag{4.14}$$

and noting that, if $F_z = 0$

$$\hat{a}_z \times (\hat{a}_z \times \bar{F}) = (\hat{a}_z \cdot \bar{F})\hat{a}_z - (\hat{a}_z \cdot \hat{a}_z)\bar{F} = -\bar{F} \tag{4.15}$$

we may obtain

$$\bar{E}_t = -\frac{j\omega\mu}{\gamma} \hat{a}_z \times \bar{H}_t - \frac{\nabla_t E_z}{\gamma} \tag{4.16}$$

Substituting (4.14) and (4.15) into (4.13) and manipulating somewhat,

we may obtain

$$\begin{aligned} j\omega\mu\epsilon_{12}(\hat{a}_z \times \bar{H}_t) &= \frac{\omega^2\mu\epsilon_{11}\bar{H}_t}{-\gamma} - \frac{j\omega\epsilon_{11}}{\gamma} \hat{a}_z \times \nabla_t E_z \\ &+ \frac{\omega\epsilon_{12}}{\gamma} \nabla_t E_z - \gamma \bar{H}_t - \nabla_t H_z \end{aligned} \tag{4.17}$$

Taking the cross product of $\hat{a}_z$ with both sides of (4.13)

$$\nabla_t H_z + \gamma \bar{H}_t = j\omega\epsilon_{11}(\hat{a}_z \times \bar{E}_t) - \omega\epsilon_{12}\bar{E}_t \tag{4.18}$$

and substituting (4.14) and (4.15) into (4.18), we obtain

$$\left[\frac{\gamma^2 + \omega^2\mu\epsilon_{11}}{-\gamma}\right](\hat{a}_z \times \bar{H}_t) = \hat{a}_z \times \left[\nabla_t H_z - \frac{\omega\epsilon_{12}}{\gamma}\nabla_t E_z\right]$$

$$-j\frac{\omega\epsilon_{11}}{\gamma}\nabla_t E_z + j\frac{\omega^2\mu\epsilon_{12}}{\gamma}\bar{H}_t \tag{4.19}$$

At this point, it will be convenient to define

$$K^2 = \gamma^2 + \omega^2\mu\epsilon_{11} \tag{4.20}$$

$$k'^2 = \omega^2\mu\epsilon_{12} \tag{4.21}$$

from which follows

$$K^2\omega\epsilon_{12} - k'^2\omega\epsilon_{11} = \gamma^2\omega\epsilon_{12} \tag{4.22}$$

Using (4.17) and (4.19) through (4.22), we obtain

$$(K^4 - k'^4)\bar{H}_t = j\hat{a}_z \times \nabla_t\left[-\gamma k'^2 H_z + (k'^2\omega\epsilon_{12} - K^2\omega\epsilon_{11})E_z\right] \tag{4.23}$$

$$+\nabla_t\left[\gamma^2\omega\epsilon_{12}E_z + K^2\gamma H_z\right]$$

Solving (4.11) for H_t, taking the cross product of $\hat{a}_z$ with both sides and substituting in (4.13), we obtain

$$\frac{\gamma^2 \bar{E}_t}{j\omega\mu} + \frac{\gamma \nabla_t E_z}{j\omega\mu} - \hat{a}_z \times \nabla_t H_z = j\omega\epsilon_{11} \bar{E}_t + \omega\epsilon_{12} \hat{a}_z \times \bar{E}_t \tag{4.24}$$

Substituting (4.11) into (4.18) and then using (4.24) and (4.20) through (4.22) we may obtain

$$(K^4 - k'^4)\bar{E}_t = \nabla_t(-\gamma K^2 E_z + \omega\mu k'^2 H_z) + j\hat{a}_z \times \nabla_t(\omega\mu K^2 H_z + \gamma k'^2 E_z) \tag{4.25}$$

If the divergence of both sides of (4.25) is taken, the divergence of the second term on the right side of (4.25) can be shown to be zero by vector identities. We then have

$$(K^4 - k'^4)\nabla \cdot \bar{E}_t = \nabla_t^2(-\gamma K^2 E_z + \omega\mu k'^2 H_z) \tag{4.26}$$

Performing a similar operation on (4.23) yields

$$(K^4 - k'^4)\nabla \cdot \bar{H}_t = \nabla_t^2(\gamma^2 \omega \epsilon_{12} E_z - K^2 \gamma H_z) \tag{4.27}$$

We may write $\bar{D} = \underline{\underline{\epsilon}} \cdot \bar{E}$ in the following manner

$$\bar{D} = \epsilon_{11}\bar{E}_t - j\epsilon_{12}(\hat{a}_z \times \bar{E}_t) + \hat{a}_z \epsilon_{33} E_z \tag{4.28}$$

If we take the divergence of D and note that (from (4.10))

$$\hat{a}_z \cdot \nabla_t \times E_t = -j\omega\mu H_z \tag{4.29}$$

we obtain

$$\frac{\nabla \cdot \bar{D}}{\epsilon_{11}} = 0 = \frac{1}{r}\left[\frac{\partial r E_r}{\partial r} + \frac{\partial E_\theta}{\partial \theta}\right] - j\frac{\epsilon_{12}}{\epsilon_{11}}\left[j\omega\mu H_z\right] - \frac{\epsilon_{33}}{\epsilon_{11}}\gamma E_z \tag{4.30}$$

writing

$$\nabla \cdot \bar{E} = \nabla \cdot \bar{E}_t - \gamma E_z \tag{4.31}$$

and making the necessary substitutions results in

$$\nabla \cdot \bar{E}_t = -\frac{\epsilon_{12}}{\epsilon_{11}}\omega\mu H_z + \frac{\epsilon_{33}}{\epsilon_{11}}\gamma E_z \tag{4.32}$$

Let us now write

$$\nabla \cdot \bar{H} = 0 = \nabla \cdot \bar{H}_t - \gamma H_z \tag{4.33}$$

If we now substitute (4.32) and (4.33) in (4.26) and (4.27), we may, after some lengthy but routine algebra obtain

$$\nabla_t^2 H_z + \left(K^2 - k'^2\frac{\epsilon_{12}}{\epsilon_{11}}\right)H_z + \frac{\gamma\omega\epsilon_{12}\epsilon_{33}}{\epsilon_{11}}E_z = 0 \tag{4.34}$$

$$\nabla_t^2 E_z - \frac{\gamma\omega\mu\epsilon_{12}}{\epsilon_{11}}H_z + K^2\frac{\epsilon_{33}}{\epsilon_{11}}E_z = 0 \tag{4.35}$$

For brevity, let these be written as

$$\nabla_t^2 H_z + aH_z + bE_z = 0 \tag{4.36}$$

$$\nabla_t^2 E_z + cE_z + dH_z = 0 \tag{4.37}$$

In order to obtain a solution to this pair of simultaneous equations, the artiface of assuming that both E_z and H_z may be expressed as linear combinations of two other functions of r and θ shall be used. The constraints between the constants that must exist to permit a solution for the two new functions will then be determined. Having found these two functions, we may then solve for E_z and H_z and use the uniqueness theorem to state that these are the solutions.

We shall now let

$$E_z = p_1 U_1 + p_2 U_2 \tag{4.38}$$

$$H_z = q_1 U_1 + q_2 U_2 \tag{4.39}$$

Substituting these two relations in (4.36) and (4.37) and manipulating, we obtain

$$\nabla_t^2 U_1 + U_1\left[\frac{p_2(aq_1+bp_1) - q_2(cp_1+dq_1)}{p_2q_1 - p_1q_2}\right]$$

$$+ U_2\left[\frac{p_2(aq_2+bp_2) - q_2(cp_2+dq_2)}{p_2q_1 - p_1q_2}\right] = 0 \tag{4.40}$$

$$\nabla_t^2 U_2 + U_2\left[\frac{p_1(aq_2+bp_2) - q_1(cp_2+dq_2)}{p_1q_2 - p_2q_1}\right]$$

$$+ U_1\left[\frac{p_1(aq_1+bp_1) - q_1(cp_1+dq_1)}{p_1q_2 - p_2q_1}\right] = 0 \tag{4.41}$$

If we require the coefficient of U_2 in (4.40) and the coefficient of U_1 in (4.41) to be zero, we will have two equations of the form

$$\nabla_t^2 F + s^2 F = 0 \tag{4.42}$$

the solution of which may be written

$$F = \left[C_1 J_m(sr) + C_2 N_m(sr)\right] e^{jm\theta} \tag{4.43}$$

Setting these coefficients equal to zero and manipulating we may obtain

$$s_1^2 = \frac{cp_1+dq_1}{p_1} = \frac{aq_1+bp_1}{q_1} \tag{4.44}$$

$$s_2^2 = \frac{cp_2 + dq_2}{p_2} = \frac{aq_2 + bp_2}{q_2} \tag{4.45}$$

Equations (4.38), (4.39), (4.44) and (4.45) comprise a system of six equations in eight unknowns leaving two relationships which we may specify arbitrarily. We could let $p_1 = p_2 = 1$. However the choice

$$p_1 = s_1^2 \qquad\qquad p_2 = s_2^2 \tag{4.46}$$

will result in a more compact notation. Making this choice, we find that

$$s_{1,2}^2 = \frac{a + c \pm \sqrt{(a-c)^2 + 4bd}}{2} \tag{4.47}$$

where the upper sign is to be taken with subscript one. In terms of the dielectric tensor (4.5)

$$s_{1,2}^2 = \frac{1}{2\epsilon_{11}}\left[K^2(\epsilon_{11} + \epsilon_{33}) - k'^2\epsilon_{12} \pm \left[\left(K^2(\epsilon_{11} - \epsilon_{33}) - k'^2\epsilon_{12}\right)^2 - 4(\gamma^2 k'^2 \epsilon_{12}\epsilon_{33})\right]^{1/2}\right] \tag{4.48}$$

It is also found that

$$q_j = \frac{s_j^2(s_j^2 - c)}{d} \tag{4.49}$$

Before proceeding to find the field quantities themselves, it is to be noted that the method breaks down when the expression in the denominators

in (4.40) and (4.41) is zero. It can readily be shown that this occurs when

$$(a-c)^2 = -4bd \tag{4.50}$$

which yields the following requirement on γ for this condition to exist

$$\gamma = \frac{\omega^2\mu}{\epsilon_{11}-\epsilon_{33}}\left[-\epsilon_{12}\sqrt{\epsilon_{12}\epsilon_{33}} \pm \sqrt{\epsilon_{12}^3\epsilon_{33}-(\epsilon_{11}-\epsilon_{33})[\epsilon_{11}(\epsilon_{11}-\epsilon_{33})-\epsilon_{12}^2]}\right] \tag{4.51}$$

This condition will be discussed in Section five and for the present we shall only note that it occurs for certain particular combinations of the system parameters and that when it does we must examine the problem more closely.

Using (4.38), (4.39), (4.46) and (4.49), we may write

$$E_z = \sum_{i=1}^{2}\left\{A_{1,i}s_i^2 J_n(s_i r) + A_{2,i}s_i^2 N_n(s_i r)\right\} e^{jn\theta} \tag{4.52}$$

$$H_z = \sum_{i=1}^{2}\left\{A_{1,i}\frac{s_i^2(s_i^2-c)}{d} J_n(s_i r) + A_{2,i}\frac{s_i^2(s_i^2-c)}{d} N_n(s_i r)\right\} e^{jn\theta} \tag{4.53}$$

as the most general expressions for E_z and H_z with the time and z dependence suppressed. We shall now limit our consideration to the axially symmetric case.

Using (4.23) and (4.25), we may obtain the remaining field components. After some routine operations, we obtain

$$(K^4 - k'^4) H_r = \sum_{i=1}^{2} \left\{ A_{1,i} s_i^3 J_1(s_i r) \left[\frac{\gamma K^2 (s_i^2 - c)}{d} - \gamma^2 \omega \epsilon_{12} \right] + A_{2,i} s_i^3 N_1(s_i r) \left[\frac{\gamma K^2 (s_i^2 - c)}{d} - \gamma^2 \omega \epsilon_{12} \right] \right\} \quad (4.54)$$

$$(K^4 - k'^4) H_\theta = j \sum_{i=1}^{2} \left\{ A_{1,i} s_i^3 J_1(s_i r) \left[\frac{\gamma k'^2 (s_i^2 - c)}{d} - k'^2 \omega \epsilon_{12} + K^2 \omega \epsilon_{11} \right] + \right.$$

$$\left. A_{2,i} s_i^3 N_1(s_i r) \left[\frac{\gamma k'^2 (s_i^2 - c)}{d} - k'^2 \omega \epsilon_{12} + K^2 \omega \epsilon_{11} \right] \right\} \quad (4.55)$$

$$(K^4 - k'^4) E_r = \sum_{i=1}^{2} \left\{ A_{1,i} s_i^3 J_1(s_i r) \left[\gamma K^2 - \frac{\omega \mu k'^2 (s_i^2 - c)}{d} \right] + A_{2,i} s_i^3 N_1(s_i r) \left[\gamma K^2 - \frac{\omega \mu k'^2 (s_i^2 - c)}{d} \right] \right\} \quad (4.56)$$

$$(K^4 - k'^4) E_\theta = j \sum_{i=1}^{2} \left\{ A_{1,i} s_i^3 J_1(s_i r) \left[\gamma k'^2 - \frac{\omega \mu K^2 (s_i^2 - c)}{d} \right] + A_{2,i} s_i^3 N_1(s_i r) \left[\gamma k'^2 - \frac{\omega \mu K^2 (s_i^2 - c)}{d} \right] \right\} \quad (4.57)$$

$$
\left\{ s_2^2 J_0(s_2c) + j\cot\psi\, s_2^3 J_1(s_2a) \left[\gamma K^2 - \omega\mu k'^2 (s_2^2 - c) \right] \right\} \left\{ \frac{s_1^2 k^2}{\rho^2} K_1^2(\rho c) \cot\psi\, J_0(s_1c) - \frac{K_0^2(\rho c)\, s_1^3 J_1(s_1c)}{(K^4 - k'^4)} \right.
$$

$$
\left. - \frac{\omega\mu}{\rho} K_1(\rho c) K_0(\rho c) \left[\frac{s_1^2 (s_1^2 - c)}{d} J_0(s_1c) - \frac{\cot\psi}{(K^4 - k'^4)} \left(\gamma k'^2 - \frac{\omega\mu K^2 (s_1^2 - c)}{d} \right) J_1(s_1c) \right] \right\} - \tag{4.58}
$$

$$
\left\{ s_1^2 J_0(s_1c) + j\cot\psi\, s_1^3 J_1(s_1c) \left[\frac{\gamma K^2 - \omega\mu k'^2 (s_1^2 - c)}{d} \right] \right\} \left\{ \frac{s_2^2 k^2}{\rho^2} K_1^2(\rho c) \cot\psi\, J_0(s_2c) - \frac{K_0^2(\rho c)\, s_2^3 J_1(s_2c)}{(K^4 - k'^4)} \right.
$$

$$
\left. - \frac{\omega\mu}{\rho} K_1(\rho c) K_0(\rho c) \left[\frac{s_2^2 (s_2^2 - c)}{d} J_0(s_2c) \; \frac{-\cot\psi}{(K^4 - k'^4)} \left(\gamma k'^2 - \frac{\omega\mu K^2 (s_2^2 - c)}{d} \right) J_1(s_2c) \right] \right\} = 0
$$

We may now proceed to solve the boundary value problem as was done in Appendix B for Method I. Since the procedure is the same, only the results will be given and the details dispensed with.

For case I, (beam and plasma filling a helix of radius c, we obtain equation (4.58) as the determinantal relationship. Before proceding to the remaining cases it is convenient to define

$$P_i = \frac{s_i^2 - C}{d} \tag{4.59}$$

$$M_i = \frac{\gamma K^2 - \omega\mu k'^2 P_i s_i}{K^4 - k'^4} \tag{4.60}$$

$$L_i = \frac{\gamma k'^2 P_i - k'^2 \omega \epsilon_{12} + K^2 \omega \epsilon_{11}}{K^4 - k'^4} \tag{4.61}$$

The additional subscript p will be attached to denote these quantities for the plasma region. Using these parameters, the determinantal relationships for cases II, III, and IV are given by equations (4.62), and (4.63) and (4.64), respectively.

$$
\begin{vmatrix}
J_0(s_1a) & J_0(s_2a) & -J_0(s_{1p}a) & -N_0(s_{1p}a) & -J_0(s_{2p}a) & -N_0(s_{2p}a) & 0 & 0 \\
P_1J_0(s_1a) & P_2J_0(s_2a) & -P_{1p}J_0(s_{1p}a) & -P_{1p}N_0(s_{1p}a) & -P_{2p}J_0(s_{2p}a) & -P_{2p}N_0(s_{2p}a) & 0 & 0 \\
M_1J_1(s_1a) & M_2J_1(s_2a) & -M_{1p}J_1(s_{1p}a) & -M_{1p}N_1(s_{1p}a) & -M_{2p}J_1(s_{2p}a) & -M_{2p}N_1(s_{2p}a) & 0 & 0 \\
L_1J_1(s_1a) & L_2J_1(s_2a) & -L_{1p}J_1(s_{1p}a) & -L_{1p}N_1(s_{1p}a) & -L_{2p}J_1(s_{2p}a) & -L_{2p}N_1(s_{2p}a) & 0 & 0 \\
0 & 0 & J_0(s_{1p}c) & N_0(s_{1p}c) & J_0(s_{2p}c) & N_0(s_{2p}c) & -K_0(pc) & 0 \\
0 & 0 & M_{1p}J_1(s_{1p}c) & M_{1p}N_1(s_{1p}c) & M_{2p}J_1(s_{2p}c) & M_{2p}N_1(s_{2p}c) & 0 & -\frac{j\omega\mu}{p}K_1(pc) \\
0 & 0 & 0 & 0 & 0 & 0 & K_0(pc) & \frac{j\omega\mu}{p}K_1(pc)\cot\psi \\
0 & 0 & \begin{matrix} P_{1p}J_0(s_{1p}c)+ \\ L_{1p}J_1(s_{1p}c)\cot\psi \end{matrix} & \begin{matrix} P_{1p}N_0(s_{1p}c)+ \\ L_{1p}N_1(s_{1p}c)\cot\psi \end{matrix} & \begin{matrix} P_{2p}J_0(s_{2p}c)+ \\ L_{2p}J_1(s_{2p}c)\cot\psi \end{matrix} & \begin{matrix} P_{2p}N_0(s_{2p}c)+ \\ L_{2p}N_1(s_{2p}c)\cot\psi \end{matrix} & \frac{j\omega\mu}{p}K_1(pc)\cot\psi & -K_0(pc)
\end{vmatrix} = 0 \qquad (4.62)
$$

$$
\begin{vmatrix}
J_0(s_1 a) & J_0(s_2 a) & -I_0(pa) & -K_0(pa) & 0 & 0 & 0 & 0 \\
L_1 J_1(s_1 a) & L_2 J_1(s_2 a) & -\frac{j\omega\epsilon_0}{p} I_1(pa) & \frac{j\omega\epsilon_0}{p} K_1(pa) & 0 & 0 & 0 & 0 \\
P_1 J_0(s_1 a) & P_2 J_0(s_2 a) & 0 & 0 & -I_0(pa) & -K_0(pa) & 0 & 0 \\
M_1 J_1(s_1 a) & M_2 J_1(s_2 a) & 0 & 0 & \frac{j\omega\mu}{p} I_1(pa) & -\frac{j\omega\mu}{p} K_1(pa) & 0 & 0 \\
0 & 0 & I_0(pc) & K_0(pc) & 0 & 0 & -K_0(pc) & 0 \\
0 & 0 & 0 & 0 & -\frac{j\omega\mu}{p} I_1(pc) & \frac{j\omega\mu}{p} K_1(pc) & 0 & -\frac{j\omega\mu}{p} K_1(pc) \\
0 & 0 & I_0(pc) & K_0(pc) & -\frac{j\omega\mu}{p} I_1(pc)\cot\psi & \frac{j\omega\mu}{p} K_1(pc)\cot\psi & 0 & 0 \\
0 & 0 & \frac{j\omega\epsilon_0}{p} I_1(pc)\cot\psi & -\frac{j\omega\epsilon_0}{p} K_1(pc)\cot\psi & I_0(pc) & K_0(pc) & \frac{j\omega\epsilon_0}{p} K_1(pc)\cot\psi & -K_0(pc)
\end{vmatrix} = 0 \qquad (4.63)
$$

$$
\begin{vmatrix}
J_0(s_1a) & J_0(s_2a) & -J_0(s_{1p}a) & -N_0(s_{1p}a) & -J_0(s_{2p}a) & -N_0(s_{2p}a) & 0 & 0 & 0 & 0 & 0 & 0 \\
P_1J_0(s_1a) & P_2J_0(s_2a) & -P_{1p}J_0(s_{1p}a) & -P_{1p}N_0(s_{1p}a) & -P_{2p}J_0(s_{2p}a) & -P_{2p}N_0(s_{2p}a) & 0 & 0 & 0 & 0 & 0 & 0 \\
M_1J_1(s_1a) & M_2J_1(s_2a) & -M_{1p}J_1(s_{1p}a) & -M_{1p}N_1(s_{1p}a) & -M_{2p}J_1(s_{2p}a) & -M_{2p}N_1(s_{2p}a) & 0 & 0 & 0 & 0 & 0 & 0 \\
L_1J_1(s_1a) & L_2J_1(s_2a) & -L_{1p}J_1(s_{1p}a) & -L_{1p}N_1(s_{1p}a) & -L_{2p}J_1(s_{2p}a) & -L_{2p}N_1(s_{2p}a) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & J_0(s_{1p}b) & N_0(s_{1p}b) & J_0(s_{2p}b) & N_0(s_{2p}b) & -I_0(pb) & -K_0(pb) & 0 & 0 & 0 & 0 \\
0 & 0 & L_{1p}J_1(s_{1p}b) & L_{1p}N_1(s_{1p}b) & L_{2p}J_1(s_{2p}b) & L_{2p}N_1(s_{2p}b) & -\frac{j\omega\epsilon_0}{p}I_1(pb) & \frac{j\omega\epsilon_0}{p}K_1(pb) & 0 & 0 & 0 & 0 \\
0 & 0 & P_{1p}J_0(s_{1p}b) & P_{1p}N_0(s_{1p}b) & P_{2p}J_0(s_{2p}b) & P_{2p}N_0(s_{2p}b) & 0 & 0 & -I_0(pb) & -K_0(pb) & 0 & 0 \\
0 & 0 & M_{1p}J_1(s_{1p}b) & M_{1p}N_1(s_{1p}b) & M_{2p}J_1(s_{2p}b) & M_{2p}N_1(s_{2p}b) & 0 & 0 & \frac{j\omega\mu}{p}I_1(pb) & -\frac{j\omega\mu}{p}K_1(pb) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_0(pc) & K_0(pc) & 0 & 0 & -K_0(pc) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{j\omega\mu}{p}I_1(pc) & \frac{j\omega\mu}{p}K_1(pc) & 0 & -\frac{j\omega\mu}{p}K_1(pc) \\
0 & 0 & 0 & 0 & 0 & 0 & I_0(pc) & K_0(pc) & -\frac{j\omega\mu}{p}I_1(pc)\cot\psi & \frac{j\omega\mu}{p}K_1(pc)\cot\psi & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{j\omega\epsilon_0}{p}I_1(pc)\cot\psi & -\frac{j\omega\epsilon_0}{p}K_1(pc)\cot\psi & I_0(pc) & K_0(pc) & \frac{j\omega\epsilon_0}{p}K_1(pc)\cot\psi & -K_0(pc)
\end{vmatrix} = 0 \qquad (4.64)
$$

5. Comparison of the Methods

Trivelpiece's "slow wave" approximation is obviously the simplest of the three methods even though it is still so complicated that computer results are required for a clear insight into the system, a fact which decreases the value of its comparative simplicity greatly. It has several shortcomings. First, it neglects the ac magnetic fields in its basic assumption, which later has the effect of denying the possible existence of a TE type of solution. This leads to difficulties when attempting to match a field solution obtained from this method to any case where a TE solution is required, as at a helix. Equation (4.53) shows that a TE solution does exist and that it has poles, indicating that, regardless of how small the arbitrary constants, the fields associated with the TE mode will become appreciable in the vicinity of these poles; hence this approximation cannot be used in these areas. One of these poles is a zero of ϵ_{3} as given by equation (A.23) and indicates that the extension of Trivelpiece's method fails in an area not predicted by itself. Equation (3.41) also brings out a significant difference with method II.

The extension of Brewer's method appears to improve the situation somewhat at a great increase in complexity. This method, by neglecting E_{θ} in equation (3.2), in effect, neglects the effect of the TE mode upon the TM mode but does predict a TE mode which is a function of the TM. The starting assumption in the analysis is that (neglecting the $B_{r} \mathring{z}$ term in (3.2))

$$B_z \dot{r} \gg E_\theta \qquad (5.1)$$

Using the relations developed within Section 2 this requires that the ratio

$$\frac{E_\theta}{B_z \dot{r}} = \frac{k^2(\epsilon_1 - 1)\left[(\omega + j\gamma u_0)^2 - \omega_c^2\right]}{(P^2 + \gamma_r^2)\,\omega(\omega + j\gamma u_0)} \qquad (5.2)$$

be small. Examination of this relation shows that it has a second order pole at

$$\omega = -j\gamma u_0 \qquad (5.3)$$

without corresponding zeroes. It is to be noted that, for $\beta^2 \gg k^2$ and the values of ω_p and ω_{pb} usually found in traveling wave tubes the ratio (5.2) will remain small. Also, the $B_r \dot{z}$ term in (3.2) has been neglected since B_r is very small. Looking at this term again shows that $\dot{z}$ has a pole at the frequency given by (5.3). The situation is not clear and the determining factor is believed to be losses which have not been included in the analysis. Examining (5.2) further shows the possibilities for failure particularly for fast waves. However no simple statement of these conditions reveals itself.

In general, this method is believed to be an improvement over Method I but, due primarily to the complex boundary conditions, it leaves little, if anything to recommend itself in preference to method III.

Method III, the Kales' solution has several interesting features. First,

coupled modes are predicted, i.e., TE and TM modes are related to each other by a constant and neither can be zero independently at other than special combinations of the system parameters. Second, a two-fold mode degeneracy is predicted since we may take either of two values for the radial propagation constant. This aspect presents complications in the boundary value problem.

Suppose that we try to match the boundary conditions using only one of the two possible modes. We would then find that only one arbitrary constant would appear in the field equations for region I and matching boundary conditions would give us four equations in three unknowns. The problem would then be over specified and would allow solutions for, at most, particular combinations of the system parameters. By taking both modes of each degenerate pair, we may match any physically realizeable boundary conditions.

Very closely related to the above is the fact that the degenerate (s_i) modes are not orthogonal. From Churchill (12), we may state the general requirements for orthogonality as

$$a_1 U_1(a) + a_2 U_1'(a) = 0$$

$$b_1 U_1(b) + b_2 U_1'(b) = 0 \qquad (5.4)$$

where U is a solution of a Sturm-Liouville system. Since these are precisely the relations which exist between E_z and H_θ and H_z and E_θ for each of the modes individually, it is obvious that the azimuthal

modes (m) are orthogonal for the same s_i. Since each of the s_i is associated with a separate Sturm-Liouville system the degenerate (s_i) modes are not necessarily orthogonal. Evaluating an integral of the form

$$\int_a^b r J_0(\alpha r) J_1(\beta r)\, dr \tag{5.5}$$

will demonstrate that, except for unusual combinations of the constants, the modes are not orthogonal.

In summary, it may be stated that the attempted extension of a relatively simple method (Trivelpiece's) develops inconsistencies with the exact solution of the model and, until numerical results can prove to the contrary, must be used with caution. The extension of Brewer's method appears to improve the situation somewhat but the complexity of the boundary conditions deprives the method of any advantage of simplicity and hence, has little to recommend itself in favor of the Kales' solution.

The Kales' solution, while complicated in its derivation, is not more difficult to handle afterwards than any other. It has the distinct advantage of being an exact mathematical solution of the model, the only limiting assumption being the derivation of the dielectric tensor itself. It is readily demonstrated that the method fails at the poles and zeroes of the components of the dielectric tensor, and at a few other special combinations of the parameters such that the problem degenerates into a much simpler one. The failure at the singularities of the dielectric tensor is of no great improtance since the model also fails under these conditions, i.e.,

infinite fields and propagation constants are not found in nature. This method also has the advantage that it offers the possibility of extenstion to include the effect of collisions between charged particles since the conductivity of a plasma is a tensor quantity of the same form as the dielectric tensor.

APPENDIX A

DERIVATION OF TENSOR DIELECTRIC CONSTANT

It is permissible to treat a plasma as described in the introduction as an equivalent charge-free dielectric whose characteristics vary as a function of frequency. Further, the equivalent dielectric constant for a plasma in a magnetic field is a tensor quantity because the electric field vector and the displacement vector are no longer related by a single multiplicative constant. The elements of this tensor are calculated by adding the convection current density to the free space displacement current density and setting the sum equal to the displacement current of the equivalent charge free region. Two cases shall be treated, first, that of a plasma alone, and second, that of a plasma with an electron beam through passing through it.

PLASMA

Neglecting a $\rho_1 v_1$ term as a second order quantity, one obtains

$$j\omega\epsilon_0 \bar{E}_1 + \rho_0 \bar{v}_1 = j\omega \underline{\underline{\epsilon}} \cdot \bar{E}_1 \tag{A.1}$$

Using the equation of motion and neglecting effects of AC magnetic fields on electron motion

$$\frac{d\bar{v}}{dt} = \frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla)\bar{v} = -\frac{e}{m}(\bar{E} + \bar{v} \times \bar{B}) \tag{A.2}$$

the following are then obtained.

$$j\omega v_{1r} = -\frac{e}{m}E_{1r} - \omega_c v_{1\theta} \qquad (A.3)$$

$$j\omega v_{1\theta} = -\frac{e}{m}E_{1\theta} + \omega_c v_{1r} \qquad (A.4)$$

$$j\omega v_{1z} = -\frac{e}{m}E_{1z} \qquad (A.5)$$

Solving for the components of velocity yields

$$v_{1r} = -\eta\left[\frac{-j\omega E_{1r}}{\omega^2 - \omega_c^2} + \frac{\omega_c E_{1\theta}}{\omega^2 - \omega_c^2}\right] \qquad (A.6)$$

$$v_{1\theta} = -\eta\left[\frac{-j\omega E_{1\theta}}{\omega^2 - \omega_c^2} + \frac{\omega_c E_{1r}}{\omega^2 - \omega_c^2}\right] \qquad (A.7)$$

$$v_{1z} = -\eta\frac{E_{1z}}{j\omega} \qquad (A.8)$$

Substituting these components of velocity in equation (A.1) and solving for the tensor,

$$\underline{\underline{\epsilon}} = \epsilon_0 \begin{Vmatrix} \epsilon_{11} & j\epsilon_{12} & 0 \\ -j\epsilon_{12} & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{Vmatrix} \qquad (A.9)$$

Where

$$\epsilon_{11} = 1 + \frac{\omega_p^2}{\omega_c^2 - \omega^2} \tag{A.10}$$

$$\epsilon_{12} = \frac{\omega_c}{\omega}\left[\frac{\omega_p^2}{\omega_c^2 - \omega^2}\right] \tag{A.11}$$

$$\epsilon_{33} = 1 - \frac{\omega_p^2}{\omega^2} \tag{A.12}$$

PLASMA AND ELECTRON BEAM

Assuming that the interaction between beam electrons and plasma electrons takes place only through the electric field (in keeping with the model described in the introduction), one may solve the force equation, and equation (A.2) for the beam electrons and the plasma electrons separately. Equations (A.6), (A.7), and (A.8) are the solutions for the plasma electrons. For the beam electrons, assuming all ac quantities vary as $e^{j\omega t - \gamma z}$ the force equation may be written

$$\frac{d\bar{v}}{dt} = j(\omega + j\gamma u_0)\bar{v}_1 = -\frac{e}{m}\left[\bar{E} + \bar{v} \times \bar{B}\right] \tag{A.13}$$

from which, the following are obtained

$$j(\omega + j\gamma u_0)v_{1r} = -\frac{e}{m}E_{1r} - \omega_c v_{1\theta} \tag{A.14}$$

$$j(\omega + j\gamma u_0) v_{1\theta} = -\frac{e}{m} E_{1\theta} + \omega_c v_{1r} \tag{A.15}$$

$$j(\omega + j\gamma u_0) v_{1z} = -\frac{e}{m} E_{1z} \tag{A.16}$$

Solving for the components of the ac velocity yields

$$v_{1r} = -\eta \left[\frac{-j(\omega + j\gamma u_0) E_{1r}}{(\omega + j\gamma u_0)^2 - \omega_c^2} + \frac{\omega_c E_{1\theta}}{(\omega + j\gamma u_0)^2 - \omega_c^2} \right] \tag{A.17}$$

$$v_{1\theta} = -\eta \left[\frac{-j(\omega + j\gamma u_0)}{(\omega + j\gamma u_0)^2 - \omega_c^2} - \frac{\omega_c E_{1r}}{(\omega + j\gamma u_0)^2 - \omega_c^2} \right] \tag{A.18}$$

$$v_{1z} = -\eta \frac{E_{1z}}{j(\omega + j\gamma u_0)} \tag{A.19}$$

The convection current density is now written as the sum of the current densities due to the beam and plasma electrons and one then obtains

$$j\omega \epsilon_0 E_1 + \rho_0 \bar{v}_1 + \rho_{0b} \bar{v}_{1b} = j\omega \underline{\underline{\epsilon}} \cdot E_1 \tag{A.20}$$

Solving for the components of the tensor

$$\underline{\underline{\epsilon}} = \epsilon_0 \begin{Vmatrix} \epsilon_1 & j\epsilon_2 & 0 \\ -j\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{Vmatrix} \tag{A.21}$$

where

$$\epsilon_1 = 1 + \frac{\omega_p^2}{\omega_c^2 - \omega^2} + \frac{\omega_{pb}^2}{\omega_c^2 - (\omega + j\gamma u_0)^2} \tag{A.22}$$

$$\epsilon_2 = \frac{\omega_c}{\omega}\left[\frac{\omega_p^2}{\omega_c^2 - \omega^2}\right] + \frac{\omega_c}{(\omega + j\gamma u_0)}\left[\frac{\omega_{pb}^2}{\omega_c^2 - (\omega + j\gamma u_0)^2}\right] \tag{A.23}$$

$$\epsilon_3 = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_{pb}^2}{(\omega + j\gamma u_0)^2} \tag{A.24}$$

APPENDIX B

SOLUTION OF BOUNDARY VALUE PROBLEM, METHOD I

CASE I (Region I and IV)

Extracting the appropriate field components from Section two, we have

Region I

$$E_z = A_1 J_0(T_1 r) \qquad H_z = A_2 I_0(pr)$$

$$H_\theta = \frac{j\omega\epsilon_0\epsilon_r}{\gamma} A_1 T_1 J_1(T_1 r) \qquad E_\theta = -A_2 \frac{j\omega\mu}{p} I_1(pr) \tag{B.1}$$

Region IV

$$E_z = B_1 K_0(pr) \qquad H_z = B_2 K_0(pr)$$

$$H_\theta = -B_1 \frac{j\omega\epsilon_0}{p} K_1(pr) \qquad E_\theta = B_2 \frac{j\omega\mu}{p} K_1(pr) \tag{B.2}$$

With the boundary conditions at the helix (r = c)

$$\begin{aligned} E_z^i &= E_z^e \\ E_\theta^i &= E_\theta^e \\ E_z^i &= -E_\theta^e \cot\psi \\ H_z^i + H_\theta^i \cot\psi &= H_z^e + H_\theta^e \cot\psi \end{aligned} \tag{B.3}$$

Matching these boundary conditions

$$A_1 \gamma J_0(T_1 c) = B_1 K_0(pc) \tag{B.4}$$

$$-A_2 \frac{j\omega\mu}{p} I_1(pc) = B_2 \frac{j\omega\mu}{p} K_1(pc) \tag{B.5}$$

$$A_1 \gamma J_0(T_1 c) = -\cot\psi \left[-A_2 \frac{j\omega\mu}{p} I_1(pc) \right] \quad . \tag{B.6}$$

$$A_2 I_0(pc) + \cot\psi \left[j\frac{\omega\epsilon_0\epsilon_1}{\gamma} A_1 T_1 J_1(T_1 c) \right] =$$

$$B_2 K_0(pc) + \cot\psi \left[-B_1 \frac{j\omega\epsilon_0}{p} K_1(pc) \right] \tag{B.7}$$

Solving (B.4), (B.5), and (B.6) for A_1, B_1, and A_2 and substituting in (B.7); one obtains, after some manipulation

$$\frac{1}{pc\,K_1(pc)} + \frac{\cot^2\psi\, k^2 I_1(pc)}{p^2} \left[\frac{K_1(pc)}{K_0(pc)} - \frac{p\epsilon_1 T_1 J_1(T_1 c)}{\gamma^2 J_0(T_1 c)} \right] = 0 \tag{B.8}$$

Making the approximation that

$$p^2 = -(\gamma^2 + k^2) \approx -\gamma^2 \qquad p \approx -j\gamma$$

and using equation (2.10), the determinantal relationship

$$\frac{1}{pc\,K_1(pc)} + \frac{\cot^2\psi\, k^2 I_1(pc)}{p^2} \left[\frac{K_1(pc)}{K_0(pc)} + \frac{j\,\epsilon_1\epsilon_3 J_1(T_1 c)}{J_0(T_1 c)} \right] = 0 \tag{B.9}$$

is obtained.

<u>CASE II</u> (Region I, II, and IV)

The appropriate field components, from section two, are

Region I

$$E_z = A_1 \gamma J_0(T_1 r) \qquad H_z = A_2 I_0(pr)$$

$$H_\theta = j\frac{\omega\epsilon_0\epsilon_1}{\gamma} A_1 T_1 J_1(T_1 r) \qquad E_\theta = -A_2 j\frac{\omega\mu}{p} I_1(pr) \tag{B.10}$$

Region II

$$E_z = B_1 \gamma J_0(T_2 r) + B_2 \gamma N_0(T_2 r)$$

$$H_\theta = j\frac{\omega\epsilon_0\epsilon_{11}}{\gamma}\left[B_1 T_2 J_1(T_2 r) + B_2 T_2 N_1(T_2 r)\right]$$

$$H_z = A_2 I_0(pr) \tag{B.11}$$

$$E_\theta = -A_2 j\frac{\omega\mu}{p} I_1(pr)$$

Region IV

$$E_z = D_1 K_0(pr) \qquad H_z = D_2 K_0(pr)$$

$$H_\theta = -D_1 j\frac{\omega\epsilon_0}{p} K_1(pr) \qquad E_\theta = D_2 j\frac{\omega\mu}{p} K_1(pr) \tag{B.12}$$

Matching tangential E and H components at the boundary between the beam and the plasma (r = a)

$$A_1 \gamma J_0(T_1 a) = B_1 \gamma J_0(T_2 a) + B_2 \gamma N_0(T_2 a) \tag{B.13}$$

$$j\frac{\omega\epsilon_0\epsilon_1}{\gamma} T_1 J_1(T_1 a) = j\frac{\omega\epsilon_0\epsilon_{11}}{\gamma}\left[B_1 T_2 J_1(T_2 a) + B_2 T_2 N_1(T_2 a)\right] \tag{B.14}$$

and eliminating the A_n coefficients, one obtains

$$B_1\left[\frac{J_0(T_2a)}{J_0(T_1a)} - \frac{\epsilon_{11}T_2J_1(T_2a)}{\epsilon_1T_1J_1(T_1a)}\right] + B_2\left[\frac{N_0(T_2a)}{J_0(T_1a)} - \frac{\epsilon_{11}T_2N_1(T_2a)}{\epsilon_1T_1J_1(T_1a)}\right] = 0 \qquad (B.15)$$

At the helix, boundary conditions (B.3) apply

$$B_1\gamma J_0(T_2c) + B_2\gamma N_0(T_2c) = D_1K_0(pc) \qquad (B.16)$$

$$-A_2\frac{j\omega\mu}{p}I_1(pc) = D_2\frac{j\omega\mu}{p}K_1(pc) \qquad (B.17)$$

$$B_1\gamma J_0(T_2c) + B_2\gamma N_0(T_2c) = A_2\cot\psi\,\frac{j\omega\mu}{p}I_1(pc) \qquad (B.18)$$

$$A_2I_0(pc) + \cot\psi\left[\frac{j\omega\epsilon_0\epsilon_{11}}{\gamma}\Big(B_1T_2J_1(T_2c) + B_2T_2N_1(T_2c)\Big)\right] =$$

$$D_2K_0(pc) - D_1\cot\psi\,\frac{j\omega\epsilon_0}{p}K_1(pc) \qquad (B.19)$$

Solving the first three of these equations for D_1, A_2, and D_2 and substituting in the fourth; one obtains, after some manipulation

$$B_1\left[\frac{\gamma J_0(T_2c)I_0(pc)}{I_1(pc)}+\frac{K_0(pc)\gamma J_0(T_2c)}{K_1(pc)}-\frac{\cot^2\psi\, k^2\epsilon_{11}}{\gamma p}T_2J_1(T_2c)\right.$$

$$\left.-\frac{k^2K_1(pc)}{p^2K_0(pc)}\gamma J_0(T_2c)\cot^2\psi\right]+B_2\left[\frac{\gamma N_0(T_2c)I_0(pc)}{I_1(pc)}+\frac{K_0(pc)}{K_1(pc)}\gamma N_0(T_2c)\right.$$

$$\left.-\frac{\cot^2\psi\, k^2\epsilon_{11}T_2}{\gamma p}N_1(T_2c)-\frac{k^2K_1(pc)}{p^2K_0(pc)}\gamma N_0(T_2c)\cot^2\psi\right]=0 \qquad \text{(B.20)}$$

Again making the approximation that

$$p \approx -j\gamma$$

one obtains

$$B_1\left[\frac{\gamma J_0(T_2c)}{pcI_1(pc)K_1(pc)}+\frac{\cot^2\psi\, k^2}{\gamma}\left(-j\sqrt{\epsilon_{11}\epsilon_{33}}\,J_1(T_2c)+\frac{K_1(pc)}{K_0(pc)}J_0(T_2c)\right)\right]+$$

$$\text{(B.21)}$$

$$B_2\left[\frac{\gamma N_0(T_2c)}{pcI_1(pc)K_1(pc)}+\frac{\cot^2\psi\, k^2}{\gamma}\left(-j\sqrt{\epsilon_{11}\epsilon_{33}}\,N_1(T_2c)+\frac{K_1(pc)}{K_0(pc)}N_0(T_2c)\right)\right]=0$$

Setting the determinant of coefficients of B_1 and B_2 obtained from equations (B.15) and (B.21) equal to zero, the determinantal relationship is then obtained.

$$\left[\frac{-jJ_0(T_2c)}{c\,I_1(pc)K_1(pc)} + \frac{\cot^2\psi\, k^2}{\gamma}\left(-j\sqrt{\epsilon_{11}\epsilon_{33}}\,J_1(T_2c) + \frac{K_1(pc)}{K_0(pc)}J_0(T_2c)\right)\right] X$$

$$\left[\frac{N_0(T_2a)}{J_0(T_1a)} - \sqrt{\frac{\epsilon_{11}\epsilon_{33}}{\epsilon_1\epsilon_3}}\frac{N_1(T_2a)}{J_1(T_2a)}\right] - \left[\frac{J_0(T_2a)}{J_0(T_1a)} - \sqrt{\frac{\epsilon_{11}\epsilon_{33}}{\epsilon_1\epsilon_3}}\frac{J_1(T_2a)}{J_1(T_1a)}\right] X$$

$$\left[\frac{-jN_0(T_2c)}{c\,I_1(pc)K_1(pc)} + \frac{\cot^2\psi\, k^2}{\gamma}\left(-j\sqrt{\epsilon_{11}\epsilon_{33}}\,N_1(T_2c) + \frac{K_1(pc)}{K_0(pc)}N_0(T_2c)\right)\right] = 0 \qquad \text{(B.22)}$$

<u>CASE III</u> (Region I, III, and IV)

The appropriate field components, extracted from Section two, are

Region I (Beam and Plasma)

$$E_z = A_1\gamma J_0(T_1r) \qquad H_z = A_2 I_0(pr)$$

$$H_\theta = j\frac{\omega\epsilon_0\epsilon_1}{\gamma}A_1T_1J_1(T_1r) \qquad E_\theta = -A_2\, j\frac{\omega\mu}{p}I_1(pr) \qquad \text{(B.23)}$$

Region II (Free space)

$$E_z = C_1 I_0(pr) + C_2 K_0(pr)$$

$$H_\theta = j\frac{\omega\epsilon_0}{p}\left[C_1 I_1(pr) - C_2 K_1(pr)\right]$$

$$H_z = C_3 I_0(pr) + C_4 K_0(pr) \qquad \text{(B.24)}$$

$$E_\theta = -j\frac{\omega\mu}{p}\left[C_3 I_1(pr) - C_4 K_1(pr)\right]$$

Region IV (Free space)

$$E_z = D_1 K_0(pr) \qquad H_z = D_2 K_0(pr)$$

$$H_\theta = -D_1\, j\frac{\omega\epsilon_0}{p}K_1(pr) \qquad E_\theta = D_2\, j\frac{\omega\mu}{p}K_1(pr) \qquad \text{(B.25)}$$

Matching tangential E and H components at the boundary between the beam and free space at r = b

$$A_2 I_0(pb) = C_3 I_0(pb) + C_4 K_0(pb) \tag{B.26}$$

$$-A_2\, j\frac{\omega\mu}{p} I_1(pb) = -j\frac{\omega\mu}{p} C_3 I_1(pb) + j\frac{\omega\mu}{p} C_4 K_1(pb) \tag{B.27}$$

From the two equations immediately preceding, the following is obtained

$$A_2 = C_3 + C_4 \frac{K_0(pb)}{I_0(pb)} \tag{B.28}$$

$$A_2 = C_3 - C_4 \frac{K_1(pb)}{I_1(pb)} \tag{B.29}$$

It is then concluded that C_4 is zero and that $A_2 = C_3$. Matching E_z and H_θ

$$A_1 \gamma J_0(T_1 b) = C_1 I_0(pb) + C_2 K_0(pb) \tag{B.30}$$

$$j\frac{\omega\epsilon_0\epsilon_1}{\gamma} A_1 T_1 J_1(T_1 b) = \frac{j\omega\epsilon_0}{p}\left[C_1 I_1(pb) - C_2 K_1(pb)\right] \tag{B.31}$$

and eliminating A_1, the following relation is obtained

$$C_1\left[\frac{I_0(pb)}{J_0(T_1 b)} - \frac{\gamma^2}{p\epsilon_1 T_1}\frac{I_1(pb)}{J_1(T_1 b)}\right] + C_2\left[\frac{K_0(pb)}{J_0(T_1 b)} + \frac{\gamma^2 K_1(pb)}{p\epsilon_1 T_1 J_1(T_1 b)}\right] = 0 \tag{B.32}$$

At the helix ($r = c$), boundary conditions (B.3) apply

$$C_1 I_0(pc) + C_2 K_0(pc) = D_1 K_0(pc) \tag{B.33}$$

$$-\frac{j\omega\mu}{p} C_3 I_1(pc) = \frac{j\omega\mu}{p} K_1(pc) \tag{B.34}$$

$$C_1 I_0(pc) + C_2 K_0(pc) = \frac{j\omega\mu}{p} C_3 I_1(pc) \cot\psi \tag{B.35}$$

$$C_3 I_0(pc) + \cot\psi \left[\frac{j\omega\epsilon_0}{p} C_1 I_1(pc) - \frac{j\omega\epsilon_0}{p} C_2 K_1(pc)\right] =$$

$$D_2 K_0(pc) + \cot\psi \left[-D_1 K_1(pc) \frac{j\omega\epsilon_0}{p}\right] \tag{B.36}$$

Eliminating D_1, D_2, and C_3, the following relation remains (after considerable manipulation)

$$C_1 \left[\frac{I_0(pc)}{I_1(pc) K_1(pc)} - \frac{k^2 \cot^2\psi}{p^2 K_0(pc)}\right] + C_2 \left[\frac{K_0(pc)}{I_1(pc) K_1(pc)}\right] = 0$$

(B.37)

Setting the determinant of coefficients of the C_n obtained from equations (B.32) and (B.37) equal to zero, making the approximation that $p \approx -j\gamma$

and using the relations between the T_n and γ, the following determinantal relation is obtained.

$$\left[\frac{I_0(pc)}{I_1(pc)K_1(pc)}\right]\left[\frac{I_0(pb)}{J_0(T_1b)} - \frac{jI_1(pb)}{\sqrt{\epsilon_1\epsilon_3}\,J_1(T_1b)}\right] - \tag{B.38}$$

$$\left[\frac{I_0(pc)}{I_1(pc)K_1(pc)} - \frac{k^2\cot^2\psi}{p^2K_0(pc)}\right]\left[\frac{K_0(pb)}{J_0(T_1b)} + \frac{jK_1(pb)}{\sqrt{\epsilon_1\epsilon_3}\,J_1(T_1b)}\right] = 0$$

CASE IV (Region I, II, III, and IV)

The field components are as given in Section two for all regions. At $r = a$, the boundary between beam and plasma, the problem is the same as for Case II. The result is

$$B_1\left[\frac{J_0(T_2a)}{J_0(T_1a)} - \sqrt{\frac{\epsilon_{11}\epsilon_{33}}{\epsilon_1\epsilon_3}}\,\frac{J_1(T_2a)}{J_1(T_1a)}\right] + \tag{B.39}$$

$$B_2\left[\frac{N_0(T_2a)}{J_0(T_1a)} - \sqrt{\frac{\epsilon_{11}\epsilon_{33}}{\epsilon_1\epsilon_3}}\,\frac{N_1(T_2a)}{J_1(T_1a)}\right] = 0$$

At $r = b$, the boundary between plasma and free space

$$B_1\gamma J_0(T_2b) + B_2\gamma N_0(T_2b) = C_1I_0(pb) + C_2K_0(pb) \tag{B.40}$$

$$\frac{\epsilon_{11}T_2}{\gamma}\left[B_1J_1(T_2b) + B_2N_1(T_2b)\right] = \frac{1}{p}\left[C_1I_1(pb) - C_2K_1(pb)\right] \tag{B.41}$$

Substituting equation (B.39) in the above and defining

$$G = J_0(T_2 a) J_1(T_1 a) - \sqrt{\frac{\epsilon_{11}\epsilon_{33}}{\epsilon_1 \epsilon_3}}\, J_1(T_2 a) J_0(T_1 a) \tag{B.42}$$

$$H = N_0(T_2 a) J_1(T_1 a) - \sqrt{\frac{\epsilon_{11}\epsilon_{33}}{\epsilon_1 \epsilon_3}}\, N_1(T_2 a) J_0(T_1 a) \tag{B.43}$$

After some manipulation, the following is obtained

$$C_1\Big[I_0(pb)\,\epsilon_{11} p T_2\big(N_1(T_2 b)G - J_1(T_2 b)H\big) - \gamma^2 I_1(pb)\big(N_0(T_2 b)G - J_0(T_2 b)H\big)\Big] + C_2\Big[K_0(pb)\,\epsilon_{11} p T_2\big(N_1(T_2 b)G - J_1(T_2 b)H\big) + \gamma^2 K_1(pb)\big(N_0(T_2 b)G - J_0(T_2 b)H\big)\Big] = 0 \tag{B.44}$$

At the helix (r = c), the situation is the same as for Case III. We then have

$$C_1\left[\frac{I_0(pc)}{I_1(pc)} - \frac{k^2 \cot^2\psi\, K_1(pc)}{p^2 K_0(pc)}\right] + C_2\left[\frac{K_0(pc)}{I_1(pc)}\right] = 0 \tag{B.45}$$

If the determinant of coefficients of the C_n obtained form equations (B.44) and (B.45) is set equal to zero and the approximation that $p \approx -j\gamma$ is made and the definitions of the T_n used, the determinantal relationship may be expressed as

$$\left[-j\sqrt{\epsilon_{33}\,\epsilon_{11}}\,K_0(\rho b)\Big(N_1(T_2 b)G - J_1(T_2 b)H\Big) + K_1(\rho b)\Big(N_0(T_2 b)G - J_0(T_2 b)H\Big)\right] \times$$

$$\left[\frac{I_0(\rho c)}{I_1(\rho c)} - \frac{k^2 \cot^2\psi}{\rho^2}\frac{K_1(\rho c)}{K_0(\rho c)}\right] - \left[\frac{K_0(\rho c)}{I_1(\rho c)}\right] \times \qquad (B.46)$$

$$\left[-j\sqrt{\epsilon_{33}\,\epsilon_{11}}\,I_0(\rho b)\Big(N_1(T_2 b)G - J_1(T_2 b)H\Big) - I_1(\rho b)\Big(N_0(T_2 b)G - J_0(T_2 b)H\Big)\right] = 0$$

APPENDIX C

SOLUTION OF BOUNDARY VALUE PROBLEMS, METHOD II

Case I (Region I and IV)

Extracting the appropriate field quantities from Section three

Region I (Beam and Plasma, $0 \le r \le c$)

$$E_z = A_1 J_0(\gamma_n c) \tag{C.1}$$

$$H_\theta = \frac{-\gamma_n}{j\omega\mu}\left[1 + \frac{\gamma^2}{p_h^2}\right] A_1 J_1(\gamma_n c) \tag{C.2}$$

$$H_z = A_1 F_2 J_0(\gamma_n c) + A_3 I_0(p_h c) \tag{C.3}$$

$$E_\theta = \frac{j\omega\mu\gamma\gamma_n\omega_c\epsilon_0(\epsilon_1 - 1)}{p_h^2(p_h^2 + \gamma_n^2)} A_1 J_1(\gamma_n c) - \frac{j\omega\mu}{p_h} A_3 I_1(p_h c) \tag{C.4}$$

Region IV (Free Space, $r \ge c$)

$$E_z = D_1 K_0(pc) \tag{C.5}$$

$$H_\theta = -\frac{j\omega\epsilon_0}{p} D_1 K_1(pc) \tag{C.6}$$

$$H_z = D_2 K_0 (pc) \tag{C.7}$$

$$E_\theta = \frac{j\omega\mu}{p} D_2 K_1 (pc) \tag{C.8}$$

The boundary conditions are derived from the basic conditions (3.74) and the boundary requirements of the "sheath helix" model. Brief comment has already been made that this case must be considered as the limiting situation of what amounts to case three (to be considered) with the free space region shrinking to zero thickness. If we view the problem in this light, the boundary conditions are found to be

$$E_{z_I} = E_{z_{II}} \tag{C.9}$$

$$E_{\theta_I} = E_{\theta_{II}} \tag{C.10}$$

$$E_{z_I} = -E_{\theta_I} \cot \psi \tag{C.11}$$

$$H_{z_I} - G_\theta + (H_{\theta_I} + G_z) \cot \psi = H_{z_{II}} + H_{\theta_{II}} \cot \psi \tag{C.12}$$

Forming the quantities required by the last two relations, using (3.26),

(3.65), (3.66) and the relations given above for region I, we obtain

$$H_z - G_\theta = A_1 \left[\frac{\omega_c c \epsilon_0 (\epsilon_1 - 1) \gamma \gamma_r}{2 p_h^2} J_1(\gamma_r c) + F_2 J_0(\gamma_r c) \right] + A_3 I_0(p_h c) \tag{C.13}$$

$$H_\theta + G_z = A_1 \left[\frac{-\gamma_r}{j\omega\mu p_h^2} \left(\frac{j\gamma u_0 k^2}{\omega} (\epsilon_1 - \epsilon_{11}) + p_h^2 + \gamma^2 \right) J_1(\gamma_r c) \right] \tag{C.14}$$

Applying the boundary conditions given above, we obtain

$$D_1 K_0(pc) = A_1 J_0(\gamma_r c) \tag{C.15}$$

$$\frac{j\omega\mu \gamma\gamma_r \omega_c \epsilon_0 (\epsilon_1 - 1)}{p_h^2 (p_h^2 + \gamma_r^2)} A_1 J_1(\gamma_r c) - \frac{j\omega\mu}{p_h} A_3 I_1(p_h c) = \frac{j\omega\mu}{p} K_1(pc) \tag{C.16}$$

$$D_1 K_0(pc) = -D_2 \frac{j\omega\mu}{p} \cot\psi \, K_1(pc) \tag{C.17}$$

$$\begin{aligned} &A_1 \left[\frac{\omega_c c \epsilon_0 (\epsilon_1 - 1) \gamma\gamma_r}{2 p_h^2} J_1(\gamma_r c) + F_2 J_0(\gamma_r c) \right] + A_3 I_0(p_h c) + \\ &A_1 \left[\frac{-\gamma_r}{j\omega\mu p_h^2} \left(\frac{j\gamma u_0 k^2}{\omega} (\epsilon_1 - \epsilon_{11}) + p_h^2 + \gamma^2 \right) J_1(\gamma_r c) \right] \cot\psi = \\ &D_2 K_0(pc) - D_1 \frac{j\omega\epsilon_0}{p} K_1(pc) \cot\psi \end{aligned} \tag{C.18}$$

Using the same method as used in Appendix B, the determinantal relationship is found to be

$$\left[\frac{j\omega\mu\gamma\gamma_n\omega_c\epsilon_0(\epsilon_1-1)J_1(\gamma_n c)}{\beta_h^2(\beta_h^2+\gamma_n^2)} + \frac{J_0(\gamma_n c)}{\cot\psi}\right] I_0(\beta_h c) \quad +$$

$$\frac{j\omega\mu I_1(\beta_h c)}{\beta_h}\left[\frac{\gamma\gamma_n\omega_c c\epsilon_0(\epsilon_1-1)J_1(\gamma_n c)}{2\beta_h^2} + F_2 J_0(\gamma_n c)\right. \tag{C.19}$$

$$\frac{-\gamma_n}{j\omega\mu\beta_h^2}\left(j\frac{\gamma u_0 k^2(\epsilon_1-\epsilon_{11})}{\omega} + \beta_h^2 + \gamma^2\right) J_1(\gamma_n c)\cot\psi \quad +$$

$$\left.\frac{p J_0(\gamma_n c)K_0(pc)}{j\omega\mu\cot\psi\, K_1(pc)} + j\frac{\omega\epsilon_0\cot\psi\, J_0(\gamma_n c)K_1(pc)}{p K_0(pc)}\right] = 0$$

It is believed obvious, at this point, that the complexity of this method is considerably greater than that of Method I. It is also believed that nothing is to be gained by writing lengthy, complex expressions from which no insight into the nature of the problem can be obtained by inspection. The method of presentation of the determinantal relationships will, therefore, be changed to expressing the relationship in the form of a determinant set equal to zero.

<u>Case II</u> (Region I, II, and IV)

Extracting the appropriate field quantities from Section three

Region I (Beam and Plasma, $0 \leq r \leq a$)

$$E_z = A_1 J_0(\gamma_r r) \tag{C.20}$$

$$H_\theta = -\frac{\gamma_r}{j\omega\mu}\left[1+\frac{\gamma^2}{P_h^2}\right]A_1 J_1(\gamma_r r) \tag{C.21}$$

$$H_z = A_1 F_2 J_0(\gamma_r r) + A_3 I_0(P_h r) \tag{C.22}$$

$$E_\theta = \frac{j\omega\mu\,\gamma\gamma_r\omega_c\epsilon_0(\epsilon_1 - 1)}{P_h^2(P_h^2+\gamma_r^2)}A_1 J_1(\gamma_r r) - j\frac{\omega\mu}{P_h}A_3 I_1(P_h r) \tag{C.23}$$

Region II (Plasma, $a \leq r \leq c$)

$$E_z = B_1 J_0(\gamma_{rp} r) + B_2 N_0(\gamma_{rp} r) \tag{C.24}$$

$$H_\theta = -\frac{\gamma_{rp}}{j\omega\mu}\left[1+\frac{\gamma^2}{P_{hp}^2}\right]\left[B_1 J_1(\gamma_{rp} r) + B_2 N_1(\gamma_{rp} r)\right] \tag{C.25}$$

$$H_z = B_1 F_{2p} J_0(\gamma_{rp} r) + B_2 F_{2p} N_0(\gamma_{rp} r) + B_3 I_0(P_{hp} r) + B_4 K_0(P_{hp} r) \tag{C.26}$$

$$E_\theta = j\frac{\omega\mu\,\gamma\gamma_{rp}\,\omega_c\,\epsilon_0(\epsilon_{11}-1)}{p_{hp}^2(p_{hp}^2+\gamma_{rp}^2)}\left[B_1 J_1(\gamma_{rp} r) + B_2 N_1(\gamma_{rp} r)\right]$$

$$-j\frac{\omega\mu}{p_{hp}}\left[B_3 I_1(p_{hp} r) - B_4 K_1(p_{hp} r)\right] \qquad (C.27)$$

Region IV (Free Space, r ≥ c)

$$E_z = D_1 K_0(pc) \qquad (C.28)$$

$$H_\theta = -D_1\, j\frac{\omega\epsilon_0}{p} K_1(pc) \qquad (C.29)$$

$$H_z = D_2 K_0(pc) \qquad (C.30)$$

$$E_\theta = D_2\, j\frac{\omega\mu}{p} K_1(pc) \qquad (C.31)$$

The boundary conditions for this case must be examined. At the helix (r = 0), the same considerations apply as were discussed in case I with the simplification that G_z is zero. At the boundary between the beam and the plasma, the situation is more complex. The boundary conditions expressed as equations (3.74) apply, but we must examine the expression for G_θ. We can write an expression similar to (3.66) for the beam and also one for the rippled inner surface of the plasma region. Somewhat heuristically, the surface current arises due to the difference in the two media. Therefore, if we

let the difference between the two media approach zero in a smooth manner, this surface current must also approach zero. G_θ must then be

$$G_\theta = \frac{\omega_c a \epsilon_0(\epsilon_I - 1)}{2} E_{r_I}(a) - \frac{\omega_c a \epsilon_0(\epsilon_{II} - 1)}{2} E_{r_{II}}(a) \tag{C.32}$$

After some manipulation, we obtain at $r = a$

$$H_{\theta_I} + G_z = A_1\left[\frac{-\gamma_r}{j\omega\mu\, p_h^2}\left(\frac{j\gamma u_0 k^2(\epsilon_I - \epsilon_{II})}{\omega} + p_h^2 + \gamma^2\right) J_1(\gamma_r a)\right] \tag{C.33}$$

$$H_z - G_\theta = A_1\left[F_2 J_0(\gamma_r a) + \frac{\gamma\gamma_r \omega_c a \epsilon_0(\epsilon_I - 1)}{2 p_h^2} J_1(\gamma_r a)\right] +$$

$$A_3 I_0(p_h a) + B_1 \frac{\gamma\gamma_{rp}\omega_c a(\epsilon_{II} - 1)\epsilon_0}{2 p_{hp}^2} J_1(\gamma_{rp} a) + \tag{C.34}$$

$$B_2 \frac{\gamma\gamma_{rp}\omega_c a(\epsilon_{II} - 1)\epsilon_0}{2 p_{hp}^2} N_1(\gamma_{rp} a)$$

and at $r = c$

$$H_z - G_\theta = B_1\left[F_{2p} J_0(\gamma_{rp} c) + \frac{\gamma\gamma_{rp}\omega_c c\,\epsilon_0(\epsilon_{II} - 1)}{2 p_{hp}^2} J_1(\gamma_{rp} c)\right] + \tag{C.35}$$

$$B_2\left[F_{2p} N_0(\gamma_{rp} c) + \frac{\gamma\gamma_{rp}\omega_c c\,\epsilon_0(\epsilon_{II} - 1)}{2 p_{hp}^2} N_1(\gamma_{rp} c)\right] +$$

$$B_3 I_0(p_{hp} c) + B_4 K_0(p_{hp} c)$$

By applying the appropriate boundary conditions at the interfaces between the regions, we may write eight equations in eight unknowns (the arbitrary constants) and write the determinantal equation as the condition for a non-trivial solution, i.e., the determinant of the matrix of coefficients must equal zero. When this is done, equation (C.36) is obtained.

Case III (Region I, III, and IV)

Region I (Beam and Plasma, $0 \le r \le a$)

$$E_z = A_1 J_0(\gamma_r r) \tag{C.37}$$

$$H_\theta = \frac{-\gamma_r}{j\omega\mu}\left[1 + \frac{\gamma^2}{P_h^2}\right] A_1 J_1(\gamma_r r) \tag{C.38}$$

$$H_z = A_1 F_2 J_0(\gamma_r r) + A_3 I_0(P_h r) \tag{C.39}$$

$$E_\theta = \frac{j\omega\mu\gamma\gamma_r \omega_c \epsilon_0(\epsilon_1 - 1)}{P_h^2(P_h^2 + \gamma_r^2)} A_1 J_1(\gamma_r r) - j\frac{\omega\mu}{P_h} A_3 I_1(P_h r) \tag{C.40}$$

$$H_z(a) - G_\theta(a) = A_1\left[\frac{\gamma\gamma_r \omega_c a \epsilon_0(\epsilon_1 - 1)}{2P_h^2} J_1(\gamma_r a) + F_2 J_0(\gamma_r a)\right] + A_3 I_0(P_h a) \tag{C.41}$$

$$
\begin{vmatrix}
J_0(\gamma_n a) & 0 & -J_0(\gamma_{np} a) & -N_0(\gamma_{np} a) & 0 & 0 & 0 & 0 \\
\frac{-\gamma_n J_1(\gamma_n a)}{j\omega\mu\rho_n^2} \times \left(j\gamma\omega_c k^2(\epsilon_1-\epsilon_{11})+\rho_n^2+\gamma^2\right) & \frac{\gamma_{np}}{j\omega\mu}\left[1+\frac{\gamma^2}{\rho_{np}^2}\right]J_1(\gamma_{np} a) & \frac{\gamma_{np}}{j\omega\mu}\left[1+\frac{\gamma^2}{\rho_{np}^2}\right]N_1(\gamma_{np} a) & 0 & 0 & 0 & 0 & 0 \\
\frac{-\gamma_n^2\gamma\omega_c\epsilon_0(\epsilon_1-1)J_0(\gamma_n a)}{\rho_n^2(\rho_n^2+\gamma_n^2)} + \frac{\gamma_n\gamma\omega_c a\epsilon_0(\epsilon_1-1)J_1(\gamma_n a)}{2\rho_n^2} & I_0(\rho_n a) & \frac{\gamma_{np}\gamma\omega_c\epsilon_0(\epsilon_{11}-1)J_1(\gamma_{np} a)}{2\rho_{np}^2} + \frac{\gamma_{np}^2\gamma\omega_c\epsilon_0(\epsilon_{11}-1)J_0(\gamma_{np} a)}{\rho_{np}^2(\rho_{np}^2+\gamma_{np}^2)} & \frac{\gamma_{np}\gamma\omega_c\epsilon_0(\epsilon_{11}-1)N_1(\gamma_{np} a)}{2\rho_{np}^2} + \frac{\gamma_{np}^2\gamma\omega_c\epsilon_0(\epsilon_{11}-1)N_0(\gamma_{np} a)}{\rho_{np}^2(\rho_{np}^2+\gamma_{np}^2)} & -I_0(\rho_n a) & -K_0(\rho_n a) & 0 & 0 \\
\frac{j\omega\mu\gamma\gamma_n\omega_c\epsilon_0(\epsilon_1-1)J_1(\gamma_n a)}{\rho_n^2(\rho_n^2+\gamma_n^2)} & -\frac{j\omega\mu}{\rho_n}I_1(\rho_n a) & -\frac{j\omega\mu\gamma\gamma_{np}\omega_c\epsilon_0(\epsilon_{11}-1)J_1(\gamma_{np} a)}{\rho_{np}^2(\rho_{np}^2+\gamma_{np}^2)} & \frac{-j\omega\mu\gamma\gamma_{np}\omega_c\epsilon_0(\epsilon_{11}-1)N_1(\gamma_{np} a)}{\rho_{np}^2(\rho_{np}^2+\gamma_{np}^2)} & \frac{j\omega\mu}{\rho_{np}}I_1(\rho_{np} a) & -\frac{j\omega\mu}{\rho_{np}}K_1(\rho_{np} a) & 0 & 0 \\
0 & 0 & J_0(\gamma_{np} c) & N_0(\gamma_{np} c) & 0 & 0 & -K_0(\rho c) & 0 \\
0 & 0 & \frac{j\omega\mu\gamma\gamma_{np}\omega_c(\epsilon_1-1)J_1(\gamma_{np} c)}{\rho_{np}^2(\rho_{np}^2+\gamma_{np}^2)} & \frac{j\omega\mu\gamma\gamma_{np}\omega_c\epsilon_0(\epsilon_{11}-1)N_1(\gamma_{np} c)}{\rho_{np}^2(\rho_{np}^2+\gamma_{np}^2)} & -\frac{j\omega\mu}{\rho_{np}}I_1(\rho_{np} c) & \frac{j\omega\mu}{\rho_{np}}K_1(\rho_{np} c) & 0 & -\frac{j\omega\mu}{\rho}K_1(\rho c) \\
0 & 0 & 0 & 0 & 0 & 0 & K_0(\rho c) & \frac{j\omega\mu}{\rho}K_1(\rho c)\cot\psi \\
0 & 0 & \frac{-\gamma_{np}^2\gamma\omega_c\epsilon_0(\epsilon_{11}-1)J_0(\gamma_{np} c)}{\rho_{np}^2(\rho_{np}^2+\gamma_{np}^2)} + \frac{\gamma\gamma_{np}\omega_c\epsilon_0(\epsilon_{11}-1)J_1(\gamma_{np} c)}{2\rho_{np}^2} - \frac{\gamma_{np}}{j\omega\mu}\left[1+\frac{\gamma^2}{\rho_{np}^2}\right]J_1(\gamma_{np} c)\cot\psi & \frac{-\gamma_{np}^2\gamma\omega_c\epsilon_0(\epsilon_{11}-1)N_0(\gamma_{np} c)}{\rho_{np}^2(\rho_{np}^2+\gamma_{np}^2)} + \frac{\gamma\gamma_{np}\omega_c\epsilon_0(\epsilon_{11}-1)N_1(\gamma_{np} c)}{2\rho_{np}^2} - \frac{\gamma_{np}}{j\omega\mu}\left[1+\frac{\gamma^2}{\rho_{np}^2}\right]N_1(\gamma_{np} c)\cot\psi & I_0(\rho_{np} c) & K_0(\rho_{np} c) & -K_0(\rho c) & \frac{j\omega\epsilon_0}{\rho}K_1(\rho c)\cot\psi
\end{vmatrix} = 0
$$

$$H_\theta(a) + G_z(a) = A_1 \frac{-\gamma_r}{j\omega\mu p_h^2}\left(j\frac{\gamma u_0 k^2(\epsilon_1 - \epsilon_{11})}{\omega} + p_h^2 + \gamma^2 \right) J_1(\gamma_r a) \tag{C.42}$$

Region III (Free Space, a $\le$ r $\le$ c)

$$E_z = C_1 I_0(pr) + C_2 K_0(pr) \tag{C.43}$$

$$H_\theta = j\frac{\omega\epsilon_0}{p}\left[C_1 I_1(pr) - C_2 K_1(pr) \right] \tag{C.44}$$

$$H_z = C_3 I_0(pr) + C_4 K_0(pr) \tag{C.45}$$

$$E_\theta = -j\frac{\omega\mu}{p}\left[C_3 I_1(pr) - C_4 K_1(pr) \right] \tag{C.46}$$

Region IV (Free Space, r $\ge$ c)

$$E_z = D_1 K_0(pr) \tag{C.47}$$

$$H_\theta = -D_1 j\frac{\omega\epsilon_0}{p} K_1(pr) \tag{C.48}$$

$$H_z = D_2 K_0(pr) \tag{C.49}$$

$$E_\theta = j\frac{\omega\mu}{p} D_2 K_1(pr) \tag{C.50}$$

Applying appropriate boundary conditions, as in Case II, the determinantal relationship is found to be equation (C.51).

Case IV (Region I, II, III, and IV)

Examining this case, we find that all the required quantities have been developed for the preceding cases, therefore equation (C.52) is given as the determinantal relationship without further comment.

$$
\begin{vmatrix}
J_0(\gamma_h a) & 0 & -I_0(pa) & -K_0(pa) & 0 & 0 & 0 & 0 \\
\frac{-\gamma_h}{j\omega\mu p_h^2} J_1(\gamma_2 a) \times \left(j\frac{\gamma u_0}{\omega} k^2(\epsilon_1-\epsilon_{11}) + p_h^2 + \gamma^2\right) & 0 & -j\frac{\omega\epsilon_0}{p} I_1(pa) & j\frac{\omega\epsilon_0}{p} K_1(pa) & 0 & 0 & 0 & 0 \\
\frac{\gamma\gamma_h u_i a \epsilon_0(\epsilon_1-1) J_1(\gamma_h a)}{2 p_h^2} - \frac{\gamma\gamma_h^2 u_i \epsilon_0(\epsilon_1-1) J_0(\gamma_h a)}{p_h^2(p_h^2+\gamma_h^2)} & I_0(p_h a) & 0 & 0 & -I_0(pa) & -K_0(pa) & 0 & 0 \\
\frac{j\omega\mu\gamma\gamma_h u_i \epsilon_0(\epsilon_1-1) J_1(\gamma_h a)}{p_h^2(p_h^2+\gamma_h^2)} & -j\frac{\omega\mu}{p_h} I_1(p_h a) & 0 & 0 & j\frac{\omega\mu}{p} I_1(pa) & -j\frac{\omega\mu}{p} K_1(pa) & 0 & 0 \\
0 & 0 & I_0(pc) & K_0(pc) & 0 & 0 & -K_0(pc) & 0 \\
0 & 0 & 0 & 0 & -j\frac{\omega\mu}{p} I_1(pc) & j\frac{\omega\mu}{p} K_1(pc) & 0 & -j\frac{\omega\mu}{p} K_1(pc) \\
0 & 0 & I_0(pc) & K_0(pc) & -j\frac{\omega\mu}{p} I_1(pc)\cot\psi & j\frac{\omega\mu}{p} K_1(pc)\cot\psi & 0 & 0 \\
0 & 0 & j\frac{\omega\epsilon_0}{p} I_1(pc)\cot\psi & -j\frac{\omega\epsilon_0}{p} K_1(pc)\cot\psi & I_0(pc) & K_0(pc) & j\frac{\omega\epsilon_0}{p} K_1(pc)\cot\psi & -K_0(pc)
\end{vmatrix}
= 0 \qquad \text{(C.5)}
$$

$$
\begin{vmatrix}
J_0(\gamma_n a) & 0 & -J_0(\gamma_{np} a) & -N_0(\gamma_{np} a) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-\gamma_n J_1(\gamma_n a)}{j\omega\mu p_h^2} \times \left[j\frac{\gamma\omega_c k^2}{\omega}(\epsilon_1-\epsilon_{11}) + p_h^2+\gamma^2 \right] & \frac{\gamma_{np}}{j\omega\mu}\left[1+\frac{\gamma^2}{p_{hp}^2}\right] J_1(\gamma_{np} a) & \frac{\gamma_{np}}{j\omega\mu}\left[1+\frac{\gamma^2}{p_{hp}^2}\right] N_1(\gamma_{np} a) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-\gamma_n^2 \gamma\omega_c\epsilon_0(\epsilon_1-1)J_0(\gamma_n a)}{p_h^2(p_h^2+\gamma_n^2)} + \frac{\gamma_n\gamma\omega_c a\epsilon_0(\epsilon_1-1)J_1(\gamma_n a)}{2 p_h^2} & I_0(p_h a) & \frac{\gamma_{np}^2\gamma\omega_c\epsilon_0(\epsilon_{11}-1)J_0(\gamma_{np} a)}{p_{hp}^2(p_{hp}^2+\gamma_{np}^2)} + \frac{\gamma_{np}\gamma\omega_c a\epsilon_0(\epsilon_{11}-1)J_1(\gamma_{np} a)}{2 p_{hp}^2} & \frac{\gamma^2\gamma\omega_c\epsilon_0(\epsilon_{11}-1)N_0(\gamma_{np} a)}{p_{hp}^2(p_{hp}^2+\gamma_{np}^2)} + \frac{\gamma_{np}\gamma\omega_c a\epsilon_0(\epsilon_{11}-1)N_1(\gamma_{np} a)}{2 p_{hp}^2} & -I_0(p_{hp} a) & -K_0(p_{hp} a) & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{j\omega\mu\gamma\gamma_n\omega_c\epsilon_0(\epsilon_1-1)J_1(\gamma_n a)}{p_h^2(p_h^2+\gamma_n^2)} & \frac{-j\omega\mu}{p_h} I_1(p_h a) & -\frac{j\omega\mu\gamma\gamma_{np}\omega_c\epsilon_0(\epsilon_{11}-1)J_1(\gamma_{np} a)}{p_{hp}^2(p_{hp}^2+\gamma_{np}^2)} & -\frac{j\omega\mu\gamma\gamma_{np}\omega_c\epsilon_0(\epsilon_1-1)N_1(\gamma_{np} a)}{p_{hp}^2(p_{hp}^2+\gamma_{np}^2)} & \frac{j\omega\mu}{p_{hp}} I_1(p_{hp} a) & -\frac{j\omega\mu}{p_{hp}} K_1(p_{hp} a) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & J_0(\gamma_{np} b) & N_0(\gamma_{np} b) & 0 & 0 & -I_0(pb) & -K_0(pb) & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-\gamma_{np}}{j\omega\mu}\left[1+\frac{\gamma^2}{p_{hp}^2}\right] J_1(\gamma_{np} b) & \frac{-\gamma_{np}}{j\omega\mu}\left[1+\frac{\gamma^2}{p_{hp}^2}\right] N_1(\gamma_{np} b) & 0 & 0 & \frac{-j\omega\epsilon_0}{p} I_1(pb) & \frac{j\omega\epsilon_0}{p} K_1(pb) & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{\gamma_{np}^2\gamma\omega_c\epsilon_0(\epsilon_{11}-1)J_0(\gamma_{np} b)}{p_{hp}^2(p_{hp}^2+\gamma_{np}^2)} + \frac{\gamma\gamma_{np}\omega_c b\epsilon_0(\epsilon_{11}-1)J_1(\gamma_{np} b)}{2 p_{hp}^2} & -\frac{\gamma_{np}^2\gamma\omega_c\epsilon_0(\epsilon_{11}-1)N_0(\gamma_{np} b)}{p_{hp}^2(p_{hp}^2+\gamma_{np}^2)} + \frac{\gamma\gamma_{np}\omega_c b\epsilon_0(\epsilon_{11}-1)N_1(\gamma_{np} b)}{2 p_{hp}^2} & I_0(p_{hp} b) & K_0(p_{hp} b) & 0 & 0 & -I_0(pb) & -K_0(pb) & 0 & 0 \\
0 & 0 & \frac{j\omega\mu\gamma\gamma_{np}\omega_c\epsilon_0(\epsilon_{11}-1)J_1(\gamma_{np} b)}{p_{hp}^2(p_{hp}^2+\gamma_{np}^2)} & \frac{j\omega\mu\gamma\gamma_{np}\omega_c\epsilon_0(\epsilon_{11}-1)N_1(\gamma_{np} b)}{p_{hp}^2(p_{hp}^2+\gamma_{np}^2)} & -\frac{j\omega\mu}{p_{hp}} I_1(p_{hp} b) & \frac{j\omega\mu}{p_{hp}} K_1(p_{hp} b) & 0 & 0 & \frac{j\omega\mu}{p} I_1(pb) & -\frac{j\omega\mu}{p} K_1(pb) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_0(pc) & K_0(pc) & 0 & 0 & -K_0(pc) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{j\omega\mu}{p} I_1(pc) & \frac{j\omega\mu}{p} K_1(pc) & 0 & -\frac{j\omega\mu}{p} K_1(pc) \\
0 & 0 & 0 & 0 & 0 & 0 & I_0(pc) & K_0(pc) & -\frac{j\omega\mu}{p} I_1(pc)\cot\psi & \frac{j\omega\mu}{p} K_1(pc)\cot\psi & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{j\omega\epsilon_0}{p} I_1(pc)\cot\psi & -\frac{j\omega\epsilon_0}{p} K_1(pc)\cot\psi & I_0(pc) & K_0(pc) & \frac{j\omega\epsilon_0}{p} K_1(pc)\cot\psi & -K_0(pc)
\end{vmatrix} = 0 \qquad (C.52)
$$

BIBLIOGRAPHY

1. A. W. TRIVELPIECE, Slow Wave Propagation in Plasma Waveguides, California Institute of Technology Technical Report 7, Electron Tube and Microwave Research Laboratory, May 1958

2. G. D. BOYD, Experiments on the Interaction of a Modulated Electron Beam with a Plasma, California Institute of Technology Technical Report 11, May 1959

3. G. D. BOYD, Power Flow and Gap Coupling to Slow Wave Plasma Modes, California Institute of Technology Technical Report 12, June 1959

4. J. R. PIERCE, Traveling Wave Tubes, D. Van Nostrand Co., Inc. 1950

5. R. G. E. HUTTER, Beam and Wave Electronics in Microwave Tubes, D. Van Nostrand Co., Inc. 1960

6. W. W. RIGROD and J. A. LEWIS, Wave Propagation Along a Magnetically Focused Cylindrical Electron Beam, Bell System Technical Journal, pp 399-416, March 1954

7. G. R. BREWER, Some Effects of Magnetic Field Strength on Space Charge Wave Propagation, Proc. I.R.E. pp 896-903 July 1956

8. M. L. KALES, Modes in Waveguides Containing Ferrites, J. Appl. Phys. 24, p 604, 1953

9. A. W. H. BECK, Space Charge Waves, Pergamon Press 1958

10. D. J. ROSE and M.CLARK, JR., Plasmas and Controlled Fusions, John Wiley and Sons, Inc. 1961

11. J. A. STRATTON, Electromagnetic Theory, McGraw Hill Book Co., Inc. 1941

12. R. V. CHURCHILL, Fourier Series and Boundary Value Problems, McGraw Hill Book Co., Inc. 1941

13. J. J. STAFFORD, Resonant Electromagnetic Modes in Gyroelectric Plasmas (Ph. D. thesis) University of Illinois, 1961

14. C. C. JOHNSON, Letter to Editor, Proc. I.E.E.E., pp 486-487 March 1963

[illegible]

[illegible]

[illegible]

[illegible]

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[illegible]

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